

# THE WEBSTER SCALAR CURVATURE FLOW ON CR SPHERE. PART I

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**ABSTRACT.** This is the first of two papers, in which we prove some properties of the Webster scalar curvature flow. More precisely, we establish the long-time existence,  $L^p$  convergence and the blow-up analysis for the solution of the flow. As a by-product, we prove the convergence of the CR Yamabe flow on the CR sphere. The results in this paper will be used to prove a result of prescribing Webster scalar curvature on the CR sphere, which is the main result of the second paper.

## 1. INTRODUCTION

Suppose  $(M, g_0)$  is a compact  $n$ -dimensional Riemannian manifold without boundary, where  $n \geq 3$ . As a generalization of Uniformization Theorem for surfaces, one wants to find a metric  $g$  conformal to  $g_0$  such that its scalar curvature  $R_g$  is constant. This is the so-called Yamabe problem, which was introduced by Yamabe in [43]. Trudinger [42] solved it when  $R_g \leq 0$ . For the case  $R_g > 0$ , Aubin [1] solved it when  $n \geq 6$  and  $M$  is not locally conformally flat. Schoen [39] solved the remaining cases, namely, when  $3 \leq n \leq 5$  or  $M$  is locally conformally flat. See the survey article [34] by Lee and Parker for more details.

A different approach has been introduced to solve the Yamabe problem. Hamilton [25] introduced the Yamabe flow, which is defined by

$$\frac{\partial}{\partial t} g(t) = -(R_{g(t)} - \bar{R}_{g(t)})g(t), \quad g(t)|_{t=0} = g_0,$$

where  $\bar{R}_{g(t)}$  is the average of the scalar curvature  $R_{g(t)}$  of  $g(t)$ . The Yamabe flow was considered by Chow [17], Ye [44], Schwetlick and Struwe [40]. Finally Brendle [5, 6] showed that the Yamabe flow exists for all time and converges to a metric of constant scalar curvature.

As a generalization of the Yamabe problem, one wants to know what function can be realized as the scalar curvature of some conformal metric. More precisely, given a function  $f$  on a compact  $n$ -dimensional Riemannian manifold  $(M, g_0)$ , can we find a metric  $g$  conformal to  $g_0$  such that  $R_g = f$ ? This prescribing scalar curvature problem has been studied extensively, even for the case when  $M$  is a surface, see [26, 32, 33]. Especially, the problem has attracted a lot of attention when  $(M, g_0)$  is the  $n$ -dimensional standard sphere  $S^n$ , which is the so-called Nirenberg's problem. See [9, 10, 11, 12, 41]. Especially, Chen and Xu [13] defined the scalar curvature

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flow, which is a natural generation of Yamabe flow above, as follows:

$$\frac{\partial}{\partial t}g(t) = (\alpha(t)f - R_{g(t)})g(t),$$

where  $\alpha(t)$  is given by

$$\alpha(t) \int_{S^n} f dV_{g(t)} = \int_{S^n} R_{g(t)} dV_{g(t)}.$$

Using the scalar curvature flow, Chen and Xu [13] proved the following:

**Theorem 1.1** (Chen-Xu). *Suppose that  $f$  is a smooth positive function on the  $n$ -dimensional sphere  $S^n$  with only non-degenerate critical points with Morse indices  $\text{ind}(f, x)$  and such that  $\Delta_{g_0}f(x) \neq 0$  at any such point  $x$ . Let*

$$m_i = \#\{x \in S^n : \nabla_{g_0}f(x) = 0, \Delta_{g_0}f(x) < 0, \text{ind}(f, x) = n - i\}.$$

*Further, suppose  $\delta_n = 2^{\frac{2}{n}}$  if  $3 \leq n \leq 4$  or  $2^{\frac{2}{n-2}}$  for  $n \geq 5$ . If there is no solution with coefficient  $k_i \geq 0$  to the system of equations*

$$m_0 = 1 + k_0, m_i = k_{i-1} + k_i \text{ for } 1 \leq i \leq n, k_n = 0,$$

*and  $f$  satisfies*

$$\max_{S^n} f / \min_{S^n} f < \delta_n,$$

*then  $f$  can be realized as the scalar curvature of some metric conformal to the standard metric  $g_0$ .*

The Yamabe problem can also be formulated in the context of CR manifold. Suppose that  $(M, \theta_0)$  is a compact strongly pseudoconvex CR manifold of real dimensional  $2n + 1$  with a given contact form  $\theta_0$ . The CR Yamabe problem is to find a contact form  $\theta$  conformal to  $\theta_0$  such that its Webster scalar curvature  $R_\theta$  is constant. This was introduced by Jerison and Lee in [31], and was solved by Jerison and Lee for the case when  $n \geq 2$  and  $M$  is not locally CR equivalent to the sphere  $S^{2n+1}$  in [29, 30, 31]. The remaining case, namely, when  $n = 1$  or  $M$  is locally CR equivalent to the sphere  $S^{2n+1}$ , was solved by Gamara and Yacoub in [22, 24].

As an analogue to Yamabe flow, one can consider the CR Yamabe flow defined by

$$\frac{\partial}{\partial t}\theta(t) = -(R_{\theta(t)} - \overline{R}_{\theta(t)})\theta(t), \quad \theta(t)|_{t=0} = \theta_0,$$

where  $\overline{R}_{\theta(t)}$  is the average of the Webster scalar curvature  $R_{\theta(t)}$  of  $\theta(t)$ . Cheng and Chang [8] proved the short time existence of the CR Yamabe flow. Zhang [45] proved the long time existence and convergence for the case  $R_{\theta_0} < 0$ . For the case  $R_{\theta_0} > 0$ , Chang, Chiu, and Wu [7] proved the convergence when  $M$  is spherical and  $n = 1$  and  $\theta_0$  is torsion-free. For general  $n$ , the author proved in [27] the long time existence for the case when  $R_{\theta_0} > 0$ , and the convergence when  $M$  is the CR sphere.

As a generalization of CR Yamabe problem, one can consider prescribing CR Webster scalar curvature problem: given a function  $f$  on a CR manifold  $(M, \theta_0)$ , we want to find a contact form  $\theta$  conformal to  $\theta_0$  such that its Webster scalar curvature  $R_\theta = f$ . This has been studied in [15, 19, 23, 27, 35, 36, 38]. As an analogy of Nirenberg's problem, we study the problem of prescribing Webster scalar curvature

on the CR sphere  $(S^{2n+1}, \theta_0)$ . Motivated by the work of Chen and Xu in [13], we define the Webster scalar curvature flow:

$$\frac{\partial}{\partial t} \theta(t) = (\alpha(t)f - R_{\theta(t)})\theta(t),$$

where  $R_{\theta(t)}$  is the Webster scalar curvature of the contact form  $\theta(t)$ , and  $\alpha(t)$  is given by

$$\alpha(t) \int_{S^{2n+1}} f dV_{\theta(t)} = \int_{S^{2n+1}} R_{\theta(t)} dV_{\theta(t)}.$$

We assume that  $f$  is a smooth positive Morse function on the CR sphere  $S^{2n+1}$  with only non-degenerate critical points in the sense that if  $f'(x) = 0$ , then the sub-Laplacian  $\Delta_{\theta_0} f(x) \neq 0$ . Here  $f'(x)$  denotes the gradient of  $f$  with respect to the standard Riemannian metric on  $S^{2n+1}$ . By using the Webster scalar curvature flow, we will prove the following theorem:

**Theorem 1.2.** *Suppose that  $n \geq 2$  and  $f$  is a smooth positive function on  $S^{2n+1}$  with only non-degenerate critical points with Morse indices  $\text{ind}(f, x)$  and such that  $\Delta_{\theta_0} f(x) \neq 0$  at any such point  $x$ . Let*

$$m_i = \#\{x \in S^{2n+1} : f'(x) = 0, \Delta_{\theta_0} f(x) < 0, \text{ind}(f, x) = 2n + 1 - i\}.$$

*If there is no solution with coefficient  $k_i \geq 0$  to the system of equations*

$$m_0 = 1 + k_0, m_i = k_{i-1} + k_i \text{ for } 1 \leq i \leq 2n + 1, k_{2n+1} = 0,$$

*and  $f$  satisfies the simple bubble condition, namely*

$$(sbc) \quad \max_{S^{2n+1}} f / \min_{S^{2n+1}} f < 2^{\frac{1}{n}},$$

*then  $f$  can be realized as the Webster scalar curvature of some contact form conformal to  $\theta_0$ .*

In this paper we first establish some results which will be used to prove Theorem 1.2, and the proof of Theorem 1.2 will be postponed to the forthcoming paper due to its length. The paper is organized as follows. In section 2, we prove some properties of the Webster scalar curvature flow, including the uniform lower bound of the Webster scalar curvature for all time  $t \geq 0$  and the long time existence. In section 3, we establish the convergence in the sense of  $S_1^2$  and  $L^p$ , where  $p \geq 1$ , with respect to the contact form at time  $t$  as  $t \rightarrow \infty$ . Section 4 is devoted to the analysis of the blow-ups of solutions. First we prove the CR analogue of the compactness theorem obtained by Schwetlick and Struwe in [40], and we apply it to show that one of the two cases must occur: either the flow itself converges in  $S_2^p$  for some  $p > 2n + 2$ , or the corresponding normalized flow  $v(t)$ , which will be introduced there, converges. In section 5, we prove the convergence of the CR Yamabe flow on the CR sphere by using the techniques we developed in this paper. See Theorem 5.3. In particular, this recovers the result of the author in [28]. In the Appendix, we prove the CR analogue of the Aubin's improvement of the Sobolev inequality. More precisely, we proved that the Folland-Stein inequality can be improved by assuming that the conformal factor satisfies the balancing condition, i.e. the center of mass is equal to zero. This is of independent interest and is used in the compactness argument in section 4.

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## 2. THE FLOW

**2.1. The flow and its basic properties.** Let  $\theta_0$  be the standard contact form on the sphere  $S^{2n+1} = \{x = (x_1, \dots, x_{n+1}) : |x|^2 = 1\} \subset \mathbb{C}^{n+1}$ , i.e.

$$\theta_0 = \sqrt{-1}(\bar{\partial} - \partial)|x|^2 = \sqrt{-1} \sum_{j=1}^{n+1} (x_j d\bar{x}_j - \bar{x}_j dx_j).$$

Then  $(S^{2n+1}, \theta_0)$  is a compact strictly pseudoconvex CR manifold of real dimension  $2n+1$ . We define a new contact form  $\theta$  which is conformal to  $\theta_0$  as follows:

$$\theta = u^{\frac{2}{n}} \theta_0, u > 0.$$

The volume form with respect to  $\theta$  is given by  $dV_\theta := \theta \wedge d\theta^n$ , and the volume of  $S^{2n+1}$  with respect to  $\theta$  is given by  $\text{Vol}(S^{2n+1}, \theta) = \int_{S^{2n+1}} dV_\theta = \int_{S^{2n+1}} u^{2+\frac{2}{n}} dV_{\theta_0}$ .

The sub-Laplacian  $\Delta_\theta$  with respect to  $\theta$  is defined as

$$\int_{S^{2n+1}} (\Delta_\theta u) \phi dV_\theta = - \int_{S^{2n+1}} \langle \nabla_\theta u, \nabla_\theta \phi \rangle_\theta dV_\theta,$$

for any smooth function  $\phi$ . Here  $\langle \cdot, \cdot \rangle_\theta$  is the Levi form of  $\theta$  and  $\nabla_\theta$  is the sub-gradient of  $\theta$ . The Folland-Stein space  $S_1^2(S^{2n+1}, \theta_0)$  is the completion of  $C^1(S^{2n+1})$  with respect to the norm

$$\|u\|_{S_1^2(S^{2n+1}, \theta_0)} = \left( \int_{S^{2n+1}} (|\nabla_{\theta_0} u|_{\theta_0}^2 + u^2) dV_{\theta_0} \right)^{\frac{1}{2}}.$$

(For more properties about the Folland-Stein space, see [20].) If  $\Delta_\theta$  is the sub-Laplacian with respect to the contact form  $\theta$ , then it can be shown that

$$(2.1) \quad \Delta_\theta \phi = u^{-(1+\frac{2}{n})} (u \Delta_{\theta_0} \phi + 2 \langle \nabla_{\theta_0} u, \nabla_{\theta_0} \phi \rangle_{\theta_0})$$

for any smooth function  $\phi$ . We refer the reader to [18] for other definitions which are not included here.

Now suppose  $f$  is a smooth positive function on  $S^{2n+1}$ . We set

$$0 < m = \min_{S^{2n+1}} f \leq f \leq \max_{S^{2n+1}} f = M.$$

Let  $u_0 \in C^\infty(S^{2n+1})$  such that

$$(2.2) \quad \int_{S^{2n+1}} u_0^{2+\frac{2}{n}} dV_{\theta_0} = \int_{S^{2n+1}} dV_{\theta_0},$$

we define the Webster scalar curvature flow as the evolution of the contact form  $\theta = \theta(t)$ ,  $t \geq 0$  as follows:

$$(2.3) \quad \frac{\partial}{\partial t} \theta = (\alpha f - R_\theta) \theta, \quad \theta|_{t=0} = u_0^{\frac{2}{n}} \theta_0,$$

where  $R_\theta$  is the Webster scalar curvature (sometimes it is called pseudo-Hermitian scalar curvature) of the contact form  $\theta$  and  $\alpha = \alpha(t)$  is given by

$$(2.4) \quad \alpha \int_{S^{2n+1}} f dV_\theta = \int_{S^{2n+1}} R_\theta dV_\theta.$$

If we write  $\theta = u^{\frac{2}{n}} \theta_0$ , then (2.3) is equivalent to the following evolution equation of the conformal factor  $u$ :

$$(2.5) \quad \frac{\partial u}{\partial t} = \frac{n}{2}(\alpha f - R_\theta)u, \quad u|_{t=0} = u_0.$$

Since  $\theta = u^{\frac{2}{n}} \theta_0$ , the Webster scalar curvature  $R_\theta$  of  $\theta$  satisfies the following CR Yamabe equation

$$(2.6) \quad R_\theta = u^{-(1+\frac{2}{n})} \left( -(2 + \frac{2}{n}) \Delta_{\theta_0} u + R_{\theta_0} u \right),$$

where  $R_{\theta_0} = n(n+1)/2$  is the Webster scalar curvature of  $\theta_0$ .

**Proposition 2.1.** *The volume of  $S^{2n+1}$  does not change under the flow. That is,  $\text{Vol}(S^{2n+1}, \theta) = \text{Vol}(S^{2n+1}, \theta_0)$  for  $t \geq 0$ .*

*Proof.* By (2.4) and (2.5), we have

$$\begin{aligned} \frac{d}{dt} \text{Vol}(S^{2n+1}, \theta) &= \frac{d}{dt} \left( \int_{S^{2n+1}} dV_\theta \right) = \frac{d}{dt} \left( \int_{S^{2n+1}} u^{2+\frac{2}{n}} dV_{\theta_0} \right) \\ &= (2 + \frac{2}{n}) \int_{S^{2n+1}} u^{1+\frac{2}{n}} \frac{\partial u}{\partial t} dV_{\theta_0} = (n+1) \int_{S^{2n+1}} (\alpha f - R_\theta) dV_\theta = 0. \end{aligned}$$

This together with (2.2) implies  $\text{Vol}(S^{2n+1}, \theta) = \text{Vol}(S^{2n+1}, \theta)|_{t=0} = \text{Vol}(S^{2n+1}, \theta_0)$  for  $t \geq 0$ .  $\square$

We define

$$(2.7) \quad E(u) = \int_{S^{2n+1}} \left( (2 + \frac{2}{n}) |\nabla_{\theta_0} u|_{\theta_0}^2 + R_{\theta_0} u^2 \right) dV_{\theta_0} = \int_{S^{2n+1}} R_\theta dV_\theta$$

where the last equality follows from (2.6). By (2.4) and (2.7), we have

$$(2.8) \quad \alpha = \frac{E(u)}{\int_{S^{2n+1}} f u^{2+\frac{2}{n}} dV_{\theta_0}}.$$

We also define

$$(2.9) \quad E_f(u) = \frac{E(u)}{(\int_{S^{2n+1}} f u^{2+\frac{2}{n}} dV_{\theta_0})^{\frac{n}{n+1}}}.$$

**Proposition 2.2.** *The functional  $E_f$  is non-increasing along the flow. Indeed,*

$$\frac{d}{dt} E_f(u) = -n \int_{S^{2n+1}} (\alpha f - R_\theta)^2 u^{2+\frac{2}{n}} dV_{\theta_0} / \left( \int_{S^{2n+1}} f u^{2+\frac{2}{n}} dV_{\theta_0} \right)^{\frac{n}{n+1}} \leq 0.$$

*Proof.* It follows from (2.5), (2.6) and (2.7) that

$$(2.10) \quad \frac{d}{dt} E(u) = 2 \int_{S^{2n+1}} \left( -(2 + \frac{2}{n}) \Delta_{\theta_0} u + R_{\theta_0} u \right) \frac{\partial u}{\partial t} dV_{\theta_0} = n \int_{S^{2n+1}} (\alpha f - R_\theta) R_\theta dV_\theta.$$

Therefore, by (2.5) and (2.10) we obtain

$$\begin{aligned}
\frac{d}{dt}E_f(u) &= \frac{d}{dt} \left( \frac{E(u)}{\left(\int_{S^{2n+1}} f u^{2+\frac{2}{n}} dV_{\theta_0}\right)^{\frac{n}{n+1}}} \right) \\
&= \frac{d}{dt}E(u) / \left(\int_{S^{2n+1}} f u^{2+\frac{2}{n}} dV_{\theta_0}\right)^{\frac{n}{n+1}} \\
&\quad - \frac{n}{n+1} E(u) \int_{S^{2n+1}} \left(2 + \frac{2}{n}\right) f u^{1+\frac{2}{n}} \frac{\partial u}{\partial t} dV_{\theta_0} / \left(\int_{S^{2n+1}} f u^{2+\frac{2}{n}} dV_{\theta_0}\right)^{\frac{n}{n+1}+1} \\
&= n \int_{S^{2n+1}} (\alpha f - R_\theta) R_\theta dV_\theta / \left(\int_{S^{2n+1}} f u^{2+\frac{2}{n}} dV_{\theta_0}\right)^{\frac{n}{n+1}} \\
&\quad - n E(u) \int_{S^{2n+1}} f (\alpha f - R_\theta) dV_\theta / \left(\int_{S^{2n+1}} f u^{2+\frac{2}{n}} dV_{\theta_0}\right)^{\frac{n}{n+1}+1} \\
&= -n \int_{S^{2n+1}} (\alpha f - R_\theta)^2 u^{2+\frac{2}{n}} dV_{\theta_0} / \left(\int_{S^{2n+1}} f u^{2+\frac{2}{n}} dV_{\theta_0}\right)^{\frac{n}{n+1}},
\end{aligned}$$

where the last equality follows from (2.8).  $\square$

By Proposition 2.2, we have for all  $t \geq 0$ , we have

$$\int_0^t \frac{\int_{S^{2n+1}} (\alpha f - R_\theta)^2 u^{2+\frac{2}{n}} dV_{\theta_0}}{\left(\int_{S^{2n+1}} f u^{2+\frac{2}{n}} dV_{\theta_0}\right)^{\frac{n}{n+1}}} dt = \frac{1}{n} \left( E_f(u)(0) - E_f(u)(t) \right) \leq \frac{1}{n} E_f(u)(0) < \infty$$

since  $E_f(u)(t) \geq 0$  by (2.7) and (2.9). By Proposition 2.1,

$$(2.11) \quad m \text{Vol}(S^{2n+1}, \theta_0) \leq \int_{S^{2n+1}} f u^{2+\frac{2}{n}} dV_{\theta_0} \leq M \text{Vol}(S^{2n+1}, \theta_0) \text{ for all } t \geq 0,$$

hence we have

$$(2.12) \quad \int_0^\infty \int_{S^{2n+1}} (\alpha f - R_\theta)^2 u^{2+\frac{2}{n}} dV_{\theta_0} dt \leq C_0$$

where  $C_0$  is a constant depend only on  $f$  and the initial data  $u_0$ . Hence, there exists a sequence  $\{t_j\}_{j=1}^\infty$  with  $t_j \rightarrow \infty$  such that

$$(2.13) \quad \int_{S^{2n+1}} (\alpha(t_j)f - R_{\theta(t_j)})^2 u^{2+\frac{2}{n}}(t_j) dV_{\theta_0} \rightarrow 0 \text{ as } j \rightarrow \infty.$$

**2.2. Uniform lower bound of the Webster scalar curvature.** We have the following:

**Lemma 2.3.** *For any  $u \in S_1^2(S^{2n+1}, \theta_0)$ , we have*

$$R_{\theta_0} \text{Vol}(S^{2n+1}, \theta_0)^{\frac{1}{n+1}} = Y(S^{2n+1}, \theta_0) \leq \frac{E(u)}{\left(\int_{S^{2n+1}} u^{2+\frac{2}{n}} dV_{\theta_0}\right)^{\frac{n}{n+1}}}.$$

Here the CR Yamabe invariant of  $(S^{2n+1}, \theta)$  defined as

$$Y(S^{2n+1}, \theta) = \inf_{0 \neq u \in S_1^2(S^{2n+1})} \frac{\int_{S^{2n+1}} \left( \left(2 + \frac{2}{n}\right) |\nabla_\theta u|_\theta^2 + R_\theta u^2 \right) dV_\theta}{\left(\int_{S^{2n+1}} u^{2+\frac{2}{n}} dV_\theta\right)^{\frac{n}{n+1}}}$$

is a conformal invariant.

This is a result of Jerison and Lee in [29] and we refer the reader to [29] for its proof. See also [21].

Therefore, we have the following bounds for  $E(u)$ :

$$(2.14) \quad R_{\theta_0} \text{Vol}(S^{2n+1}, \theta_0) \leq E(u) \leq E_f(u(0)) \left( \left( \max_{S^{2n+1}} f \right) \text{Vol}(S^{2n+1}, \theta_0) \right)^{\frac{n}{n+1}}.$$

To see this, we note that by (2.9), Proposition 2.1 and Proposition 2.2 we have

$$\begin{aligned} E(u) &= E_f(u) \left( \int_{S^{2n+1}} f u^{2+\frac{2}{n}} dV_{\theta_0} \right)^{\frac{n}{n+1}} \leq E_f(u(0)) \left( \left( \max_{S^{2n+1}} f \right) \int_{S^{2n+1}} u^{2+\frac{2}{n}} dV_{\theta_0} \right)^{\frac{n}{n+1}} \\ &= E_f(u(0)) \left( \left( \max_{S^{2n+1}} f \right) \text{Vol}(S^{2n+1}, \theta_0) \right)^{\frac{n}{n+1}}. \end{aligned}$$

On the other hand, it follows Proposition 2.1 and Lemma 2.3 that

$$E(u) \geq Y(S^{2n+1}, \theta) \left( \int_{S^{2n+1}} u^{2+\frac{2}{n}} dV_{\theta_0} \right)^{\frac{n}{n+1}} = R_{\theta_0} \text{Vol}(S^{2n+1}, \theta_0).$$

**Lemma 2.4.** *There exists positive constants  $\alpha_1$  and  $\alpha_2$ , depending only on  $f$  and the initial data, such that  $0 < \alpha_1 \leq \alpha \leq \alpha_2$  for all  $t \geq 0$ .*

*Proof.* By (2.8), (2.14) and Proposition 2.1, we have

$$\alpha = \frac{E(u)}{\int_{S^{2n+1}} f u^{2+\frac{2}{n}} dV_{\theta_0}} \leq \frac{E_f(u(0)) \left( \left( \max_{S^{2n+1}} f \right) \text{Vol}(S^{2n+1}, \theta_0) \right)^{\frac{n}{n+1}}}{\left( \min_{S^{2n+1}} f \right) \text{Vol}(S^{2n+1}, \theta_0)} := \alpha_2$$

for all  $t \geq 0$ . On the other hand, by (2.14) and Proposition 2.1, we have

$$\alpha = \frac{E(u)}{\int_{S^{2n+1}} f u^{2+\frac{2}{n}} dV_{\theta_0}} \geq \frac{R_{\theta_0} \text{Vol}(S^{2n+1}, \theta_0)}{\left( \max_{S^{2n+1}} f \right) \text{Vol}(S^{2n+1}, \theta_0)} := \alpha_1 > 0$$

for all  $t \geq 0$ .  $\square$

The following lemma is also proved in [27]:

**Lemma 2.5.** *Under the flow (2.5), the Webster scalar curvature  $R_\theta$  with respect to  $\theta$  satisfies the following evolution equation:*

$$\frac{\partial R_\theta}{\partial t} = -(n+1)\Delta_\theta(\alpha f - R_\theta) + (R_\theta - \alpha f)R_\theta,$$

where  $\Delta_\theta$  is the sub-Laplacian with respect to the contact form  $\theta$ .

*Proof.* It follows from (2.5) and (2.6) that

$$\begin{aligned} \frac{\partial R_\theta}{\partial t} &= \frac{\partial}{\partial t} \left[ u^{-(1+\frac{2}{n})} \left( -\left(2 + \frac{2}{n}\right) \Delta_{\theta_0} u + R_{\theta_0} u \right) \right] \\ &= -\left(1 + \frac{2}{n}\right) u^{-(2+\frac{2}{n})} \frac{\partial u}{\partial t} \left( -\left(2 + \frac{2}{n}\right) \Delta_{\theta_0} u + R_{\theta_0} u \right) \\ &\quad + u^{-(1+\frac{2}{n})} \left( -\left(2 + \frac{2}{n}\right) \Delta_{\theta_0} \left( \frac{\partial u}{\partial t} \right) + R_{\theta_0} \frac{\partial u}{\partial t} \right) \\ &= \left(1 + \frac{n}{2}\right) (R_\theta - \alpha f) R_\theta \\ &\quad + u^{-(1+\frac{2}{n})} \left[ -(n+1) \Delta_{\theta_0} ((\alpha f - R_\theta) u) + \frac{n}{2} R_{\theta_0} (\alpha f - R_\theta) u \right]. \end{aligned}$$

The second term of the last expression can be written as

$$\begin{aligned}
& u^{-(1+\frac{2}{n})} \left[ -(n+1) \left( (\alpha f - R_\theta) \Delta_{\theta_0} u + u \Delta_{\theta_0} (\alpha f - R_\theta) \right. \right. \\
& \quad \left. \left. + 2 \langle \nabla_{\theta_0} (\alpha f - R_\theta), \nabla_{\theta_0} u \rangle_{\theta_0} \right) + \frac{n}{2} R_{\theta_0} (\alpha f - R_\theta) u \right] \\
&= \frac{n}{2} (\alpha f - R_\theta) u^{-(1+\frac{2}{n})} \left( -\left(2 + \frac{2}{n}\right) \Delta_{\theta_0} u + R_{\theta_0} u \right) \\
& \quad - (n+1) u^{-(1+\frac{2}{n})} \left( u \Delta_{\theta_0} (\alpha f - R_\theta) + 2 \langle \nabla_{\theta_0} (\alpha f - R_\theta), \nabla_{\theta_0} u \rangle_{\theta_0} \right) \\
&= \frac{n}{2} (\alpha f - R_\theta) R_\theta - (n+1) \Delta_\theta (\alpha f - R_\theta),
\end{aligned}$$

where the last equality follows from (2.1) and (2.6). Combining all these, the assertion follows.  $\square$

The following formula follows from  $(\alpha f)_t = \alpha' f$  and Lemma 2.5:

$$(2.15) \quad (\alpha f - R_\theta)_t = (n+1) \Delta_\theta (\alpha f - R_\theta) + (\alpha f - R_\theta) R_\theta + \alpha' f.$$

**Lemma 2.6.** *There exists a constant  $\alpha_0$  such that  $\alpha' \leq \alpha_0$  for all  $t > 0$ .*

*Proof.* It follows from (2.7) and (2.8) that

$$\begin{aligned}
& \alpha' \int_{S^{2n+1}} f u^{2+\frac{2}{n}} dV_{\theta_0} + \alpha \left(2 + \frac{2}{n}\right) \int_{S^{2n+1}} f u^{1+\frac{2}{n}} \frac{\partial u}{\partial t} dV_{\theta_0} \\
&= \frac{d}{dt} E(u) = 2 \int_{S^{2n+1}} \left( -\left(2 + \frac{2}{n}\right) \Delta_{\theta_0} u + R_{\theta_0} u \right) \frac{\partial u}{\partial t} dV_{\theta_0}.
\end{aligned}$$

By using (2.5), (2.6) and (2.8), it can be rewritten as

$$(2.16) \quad \frac{\alpha'}{\alpha} E(u) + (n+1) \int_{S^{2n+1}} \alpha f (\alpha f - R_\theta) dV_\theta = n \int_{S^{2n+1}} (\alpha f - R_\theta) R_\theta dV_\theta,$$

which implies that

$$(2.17) \quad \alpha' = \frac{\alpha}{E(u)} \left[ -n \int_{S^{2n+1}} (\alpha f - R_\theta)^2 dV_\theta - \int_{S^{2n+1}} \alpha f (\alpha f - R_\theta) dV_\theta \right].$$

Applying Young's inequality, which says that  $ab \leq \frac{\epsilon}{2} b^2 + \frac{1}{2\epsilon} a^2$  for any  $\epsilon > 0$ , with  $a = \alpha f$ ,  $b = -(\alpha f - R_\theta)$  and  $\epsilon = 2n$  and integrating it over  $S^{2n+1}$  with respect to  $\theta$ , we obtain

$$- \int_{S^{2n+1}} \alpha f (\alpha f - R_\theta) dV_\theta \leq n \int_{S^{2n+1}} (\alpha f - R_\theta)^2 dV_\theta + \frac{\alpha^2}{4n} \int_{S^{2n+1}} f^2 dV_\theta.$$

Hence, we have the following estimate:

$$\alpha' \leq \frac{\alpha^3}{4nE(u)} \int_{S^{2n+1}} f^2 dV_\theta \leq \frac{\alpha_2^3 M^2 \text{Vol}(S^{2n+1}, \theta_0)}{4nE_f(u(0)) (M \text{Vol}(S^{2n+1}, \theta_0))^{\frac{n}{n+1}}} := \alpha_0,$$

where we have used (2.14), Proposition 2.1, and Lemma 2.4.  $\square$

**Lemma 2.7.** *The Webster scalar curvature of  $\theta$  satisfies the following:*

$$R_\theta - \alpha f \geq \min \left\{ R_{\theta_0} - \alpha_2 M, -\frac{1}{\alpha_1 m} (\alpha_0 M + \alpha_2^2 M^2) \right\} := \gamma$$

for all  $t \geq 0$ .



*Proof.* Define  $L = \frac{\partial}{\partial t} - (n+1)\Delta_\theta + \alpha f$ . It follows from Lemma 2.5 that

$$LR_\theta = \frac{\partial}{\partial t} R_\theta - (n+1)\Delta_\theta R_\theta + \alpha f R_\theta = -(n+1)\Delta_\theta(\alpha f) + R_\theta^2.$$

Hence, if one set  $w(t) = \alpha f + \gamma$  for  $t \geq 0$ , then we have

$$\begin{aligned} Lw &= \alpha' f - (n+1)\Delta_\theta(\alpha f) + \alpha f(\alpha f + \gamma) \\ &\leq \alpha_0 M + \alpha_2^2 M^2 + \alpha_1 m \gamma - (n+1)\Delta_\theta(\alpha f) \\ &\leq -(n+1)\Delta_\theta(\alpha f) \leq LR_\theta, \end{aligned}$$

where we have used the fact that  $\gamma < 0$ , Lemma 2.4 and 2.6. Note that  $w(0) \leq \alpha_2 M + \gamma \leq R_{\theta_0}$  by Lemma 2.4. By maximum principle,  $R_\theta - \alpha f \geq \gamma$ .  $\square$

### 2.3. Long time existence.

**Lemma 2.8.** *Given any  $T > 0$ , there exist constants  $c = c(T)$ ,  $C = C(T)$  such that*

$$c \leq u(x, t) \leq C$$

for any  $(x, t) \in S^{2n+1} \times [0, T]$ .

*Proof.* By (2.5) and Lemma 2.7, we deduce

$$\frac{\partial u}{\partial t} = \frac{n}{2}(\alpha f - R_\theta)u \leq -\frac{n}{2}\gamma u$$

for all  $t$ . This implies that  $u(t) \leq e^{-\frac{n}{2}\gamma t} \leq e^{-\frac{n}{2}\gamma T} := C(T)$  for all  $0 \leq t \leq T$ , since  $u(0) = u_0 > 0$  by (2.5).

On the other hand, denote by  $P(x) = R_{\theta_0} + \sup_{0 \leq t \leq T} \sup_{S^{2n+1}} \left[ -(\alpha f + \gamma)u^{\frac{2}{n}} \right]$ . Then by (2.6) and Lemma 2.7, we get

$$\begin{aligned} 0 &\leq (R_\theta - \alpha f - \gamma)u^{1+\frac{2}{n}} = -(2 + \frac{2}{n})\Delta_{\theta_0} u + R_{\theta_0} u - (\alpha f + \gamma)u^{1+\frac{2}{n}} \\ &\leq -(2 + \frac{2}{n})\Delta_{\theta_0} u + Pu. \end{aligned}$$

Applying Proposition A.1 in [28], we can conclude that there exists a constant  $c(T)$  such that  $\inf_{S^{2n+1}} u(t) \geq c(T)$  for all  $0 \leq t \leq T$ .  $\square$

For  $p \geq 1$ , let

$$F_p(t) = \int_{S^{2n+1}} |\alpha f - R_\theta|^p dV_\theta.$$

Then for any  $p \geq 2$ , we have

$$\begin{aligned}
(2.18) \quad & \frac{d}{dt} F_p(t) = \frac{d}{dt} \left( \int_{S^{2n+1}} |\alpha f - R_\theta|^p dV_\theta \right) \\
&= p \int_{S^{2n+1}} |\alpha f - R_\theta|^{p-2} (\alpha f - R_\theta) \frac{\partial}{\partial t} (\alpha f - R_\theta) dV_\theta + \int_{S^{2n+1}} |\alpha f - R_\theta|^p \frac{\partial}{\partial t} (dV_\theta) \\
&= p \int_{S^{2n+1}} |\alpha f - R_\theta|^{p-2} (\alpha f - R_\theta) [(n+1)\Delta_\theta(\alpha f - R_\theta) + (\alpha f - R_\theta)R_\theta] dV_\theta \\
&\quad + p\alpha' \int_{S^{2n+1}} f |\alpha f - R_\theta|^{p-2} (\alpha f - R_\theta) dV_\theta + (n+1) \int_{S^{2n+1}} |\alpha f - R_\theta|^p (\alpha f - R_\theta) dV_\theta \\
&= - (n+1)p(p-1) \int_{S^{2n+1}} |\alpha f - R_\theta|^{p-2} |\nabla_\theta(\alpha f - R_\theta)|_\theta^2 dV_\theta \\
&\quad + p \int_{S^{2n+1}} R_\theta |\alpha f - R_\theta|^p dV_\theta + p\alpha' \int_{S^{2n+1}} f |\alpha f - R_\theta|^{p-2} (\alpha f - R_\theta) dV_\theta \\
&\quad + (n+1) \int_{S^{2n+1}} |\alpha f - R_\theta|^p (\alpha f - R_\theta) dV_\theta,
\end{aligned}$$

where we have used (2.5) and (2.15).

**Lemma 2.9.** *For  $p > n+1$ , there holds*

$$(2.19) \quad \frac{d}{dt} F_p(t) + \left( \int_{S^{2n+1}} |\alpha f - R_\theta|^{\frac{p(n+1)}{n}} dV_\theta \right)^{\frac{n}{n+1}} \leq C F_p(t) + C F_p(t)^{\frac{p-n}{p-n-1}}.$$

*Proof.* It follows from (2.18) that

$$\begin{aligned}
\frac{d}{dt} F_p(t) &= -\frac{2n(p-1)}{p} \int_{S^{2n+1}} \left( \left(2 + \frac{2}{n}\right) |\nabla_\theta |\alpha f - R_\theta|^{\frac{p}{2}}|_\theta^2 + R_\theta |\alpha f - R_\theta|^p \right) dV_\theta \\
&\quad + \left(p + \frac{2n(p-1)}{p}\right) \int_{S^{2n+1}} R_\theta |\alpha f - R_\theta|^p dV_\theta \\
&\quad + (n+1) \int_{S^{2n+1}} |\alpha f - R_\theta|^p (\alpha f - R_\theta) dV_\theta + p\alpha' \int_{S^{2n+1}} f |\alpha f - R_\theta|^{p-2} (\alpha f - R_\theta) dV_\theta \\
&\leq -\frac{2n(p-1)Y(S^{2n+1}, \theta_0)}{p} \left( \int_{S^{2n+1}} |\alpha f - R_\theta|^{\frac{p(n+1)}{n}} dV_\theta \right)^{\frac{n}{n+1}} \\
&\quad + C F_p(t) + C F_{p+1}(t) + \alpha' \int_{S^{2n+1}} f |\alpha f - R_\theta|^{p-2} (\alpha f - R_\theta) dV_\theta,
\end{aligned}$$

where we have used Lemma 2.3. By (2.14), (2.17), and Hölder's inequality, we have

$$\begin{aligned}
& \alpha' \int_{S^{2n+1}} f |\alpha f - R_\theta|^{p-2} (\alpha f - R_\theta) dV_\theta \\
&= \frac{\alpha}{E(u)} \left[ -n \int_{S^{2n+1}} (\alpha f - R_\theta)^2 dV_\theta - \int_{S^{2n+1}} \alpha f (\alpha f - R_\theta) dV_\theta \right] \left( \int_{S^{2n+1}} f |\alpha f - R_\theta|^{p-2} (\alpha f - R_\theta) dV_\theta \right) \\
&\leq C \left( \int_{S^{2n+1}} |\alpha f - R_\theta|^2 dV_\theta \right) \left( \int_{S^{2n+1}} |\alpha f - R_\theta|^{p-1} dV_\theta \right) \\
&\leq C \text{Vol}(S^{2n+1}, \theta)^{\frac{p-2}{p}} \left( \int_{S^{2n+1}} |\alpha f - R_\theta|^p dV_\theta \right)^{\frac{2}{p}} \text{Vol}(S^{2n+1}, \theta)^{\frac{1}{p}} \left( \int_{S^{2n+1}} |\alpha f - R_\theta|^p dV_\theta \right)^{\frac{p-1}{p}} \\
&= CF_p(t)^{\frac{p+1}{p}} \leq \begin{cases} CF_p(t)^{\frac{p-n}{p-n-1}}, & \text{if } F_p(t) \geq 1; \\ CF_p(t), & \text{if } F_p(t) < 1. \end{cases}
\end{aligned}$$

On the other hand, for any  $0 < \epsilon < 1$ , by Hölder's inequality, we have

$$\begin{aligned}
& \int_{S^{2n+1}} |\alpha f - R_\theta|^{p+1} dV_\theta \\
&\leq \left( \int_{S^{2n+1}} |\alpha f - R_\theta|^{\frac{p(n+1)}{n}} dV_\theta \right)^{\frac{n}{p}} \left( \int_{S^{2n+1}} |\alpha f - R_\theta|^p dV_\theta \right)^{\frac{p-n}{p}} \\
&\leq \epsilon \left( \int_{S^{2n+1}} |\alpha f - R_\theta|^{\frac{p(n+1)}{n}} dV_\theta \right)^{\frac{n}{n+1}} + C(\epsilon) \left( \int_{S^{2n+1}} |\alpha f - R_\theta|^p dV_\theta \right)^{\frac{p-n}{p-n-1}}
\end{aligned}$$

where we have used Young's inequality, which says that  $ab \leq \epsilon a^{\frac{p}{n+1}} + C(\epsilon) b^{\frac{p}{p-n-1}}$  for any  $a, b \geq 0$ . Thus, if we choose  $\epsilon$  small enough, one obtains the result.  $\square$

For  $T > 0$ , let

$$\delta = \sup_{0 \leq t \leq T} \|\alpha f + \gamma\|_{C^0(S^{2n+1})} + 1$$

where  $\gamma$  is the constant given in Lemma 2.7. Then  $R_\theta + \delta \geq R_\theta - (\alpha f + \gamma) + 1 \geq 1$  by Lemma 2.7.

**Lemma 2.10.** *For  $p > 2$ , there holds*

$$\begin{aligned}
(2.20) \quad & \frac{d}{dt} \left( \int_{S^{2n+1}} (R_\theta + \delta)^p dV_\theta \right) = -\frac{4(n+1)(p-1)}{p} \int_{S^{2n+1}} |\nabla_\theta (R_\theta + \delta)|^2 dV_\theta \\
& + (n+1)p(p-1) \int_{S^{2n+1}} (R_\theta + \delta)^{p-2} \langle \nabla_\theta (R_\theta + \delta), \nabla_\theta (\alpha f + \delta) \rangle_\theta dV_\theta \\
& - (n+1-p) \int_{S^{2n+1}} (R_\theta + \delta)^p (R_\theta - \alpha f) dV_\theta \\
& - p\delta \int_{S^{2n+1}} [(R_\theta + \delta)^{p-1} - (\alpha f + \delta)^{p-1}] (R_\theta - \alpha f) dV_\theta \\
& - p\delta \int_{S^{2n+1}} (\alpha f + \delta)^{p-1} (R_\theta - \alpha f) dV_\theta.
\end{aligned}$$

*Proof.* Note that

$$\begin{aligned} \frac{d}{dt} \left( \int_{S^{2n+1}} (R_\theta + \delta)^p dV_\theta \right) &= p \int_{S^{2n+1}} (R_\theta + \delta)^{p-1} \frac{\partial R_\theta}{\partial t} dV_\theta + \int_{S^{2n+1}} (R_\theta + \delta)^p \frac{\partial}{\partial t} (dV_\theta) \\ &= p \int_{S^{2n+1}} (R_\theta + \delta)^{p-1} [-(n+1)\Delta_\theta(\alpha f - R_\theta) + (R_\theta - \alpha f)R_\theta] dV_\theta \\ &\quad + (n+1) \int_{S^{2n+1}} (R_\theta + \delta)^p (\alpha f - R_\theta) dV_\theta. \end{aligned}$$

Here we have used (2.5) and Lemma 2.5. The last expression can be written as

$$\begin{aligned} &(n+1)p \int_{S^{2n+1}} (R_\theta + \delta)^{p-1} \Delta_\theta [(R_\theta + \delta) - (\alpha f + \delta)] dV_\theta \\ &\quad + p \int_{S^{2n+1}} (R_\theta + \delta)^{p-1} (R_\theta - \alpha f)(R_\theta + \delta - \delta) dV_\theta + (n+1) \int_{S^{2n+1}} (R_\theta + \delta)^p (\alpha f - R_\theta) dV_\theta \\ &= -\frac{4(n+1)(p-1)}{p} \int_{S^{2n+1}} |\nabla_\theta (R_\theta + \delta)|_\theta^2 dV_\theta \\ &\quad + (n+1)p(p-1) \int_{S^{2n+1}} (R_\theta + \delta)^{p-2} \langle \nabla_\theta (R_\theta + \delta), \nabla_\theta (\alpha f + \delta) \rangle_\theta dV_\theta \\ &\quad - (n+1-p) \int_{S^{2n+1}} (R_\theta + \delta)^p (R_\theta - \alpha f) dV_\theta - p\delta \int_{S^{2n+1}} (R_\theta + \delta)^{p-1} (R_\theta - \alpha f) dV_\theta. \end{aligned}$$

Note that the last integral can be written as

$$\begin{aligned} \int_{S^{2n+1}} (R_\theta + \delta)^{p-1} (R_\theta - \alpha f) dV_\theta &= \int_{S^{2n+1}} [(R_\theta + \delta)^{p-1} - (\alpha f + \delta)^{p-1}] (R_\theta - \alpha f) dV_\theta \\ &\quad + \int_{S^{2n+1}} (\alpha f + \delta)^{p-1} (R_\theta - \alpha f) dV_\theta. \end{aligned}$$

Combining all these, we prove the assertion.  $\square$

**Lemma 2.11.** *For any fixed  $T > 0$ , there exists constant  $C = C(T)$  such that*

$$\int_{S^{2n+1}} |\alpha f - R_\theta|^p dV_\theta \leq C(T) \text{ for all } 0 \leq t \leq T \text{ and } n+1 < p < \frac{(n+1)^2}{n}.$$

*Proof.* Take  $p = n+1$  in (2.20), we deduce that

$$\begin{aligned} \frac{d}{dt} \left( \int_{S^{2n+1}} (R_\theta + \delta)^{n+1} dV_\theta \right) &\leq -4n \int_{S^{2n+1}} |\nabla_\theta (R_\theta + \delta)|_\theta^{\frac{n+1}{2}} dV_\theta \\ &\quad + n(n+1)^2 \int_{S^{2n+1}} (R_\theta + \delta)^{n-1} \langle \nabla_\theta (R_\theta + \delta), \nabla_\theta (\alpha f + \delta) \rangle_\theta dV_\theta \\ &\quad - (n+1)\delta \int_{S^{2n+1}} (\alpha f + \delta)^n (R_\theta - \alpha f) dV_\theta. \end{aligned}$$

Note that

$$-(n+1)\delta \int_{S^{2n+1}} (\alpha f + \delta)^n (R_\theta - \alpha f) dV_\theta \leq -(n+1)\delta \gamma \int_{S^{2n+1}} (\alpha f + \delta)^n dV_\theta \leq C$$

where we have used Proposition 2.1, Lemma 2.4, and Lemma 2.7. Note also that for every  $0 < \epsilon < 1$ , by Young's inequality and Hölder's inequality and Lemma 2.8,

one obtains

$$\begin{aligned}
& \left| \int_{S^{2n+1}} (R_\theta + \delta)^{n-1} \langle \nabla_\theta(R_\theta + \delta), \nabla_\theta(\alpha f + \delta) \rangle_\theta dV_\theta \right| \\
& \leq \epsilon \int_{S^{2n+1}} |\nabla_\theta(R_\theta + \delta)|_\theta^{\frac{n+1}{2}} dV_\theta + C(\epsilon) \int_{S^{2n+1}} |\nabla_\theta(\alpha f + \delta)|_\theta^2 (R_\theta + \delta)^{n-1} dV_\theta \\
& \leq \epsilon \int_{S^{2n+1}} |\nabla_\theta(R_\theta + \delta)|_\theta^{\frac{n+1}{2}} dV_\theta + C \left( \int_{S^{2n+1}} (R_\theta + \delta)^{n+1} dV_\theta \right)^{\frac{n-1}{n+1}}.
\end{aligned}$$

If we choose  $\epsilon = \frac{2}{(n+1)^2}$  and let  $y(t) = \int_{S^{2n+1}} (R_\theta + \delta)^{n+1} dV_\theta$ , then we have

$$(2.21) \quad \frac{d}{dt} y(t) + 2n \int_{S^{2n+1}} |\nabla_\theta(R_\theta + \delta)|_\theta^{\frac{n+1}{2}} dV_\theta \leq C y(t)^{\frac{n-1}{n+1}} + C.$$

We claim that  $y(t) \leq C(T)$  for all  $0 \leq t \leq T$ . When  $n = 1$ , it follows from (2.21) that  $\frac{dy}{dt} \leq C$ , which implies that  $y(t) \leq C(T)y(0)$  for  $0 \leq t \leq T$ . When  $n > 1$ , by (2.21) we have

$$\frac{d}{dt} \left( y(t)^{\frac{2}{n+1}} \right) = y(t)^{-\frac{n-1}{n+1}} \frac{d}{dt} y(t) \leq C + C y(t)^{-\frac{n-1}{n+1}} \leq C,$$

where the last inequality follows from  $y(t) = \int_{S^{2n+1}} (R_\theta + \delta)^{n+1} dV_\theta \geq \int_{S^{2n+1}} dV_\theta = \text{Vol}(S^{2n+1}, \theta_0)$  since  $R_\theta + \delta \geq 1$ . Hence,  $y(t)^{\frac{2}{n+1}} \leq C(T)$  for  $0 \leq t \leq T$ . This proves the claim.

Now by (2.21) and the claim, we have

$$\max_{0 \leq t \leq T} \int_{S^{2n+1}} (R_\theta + \delta)^{n+1} dV_\theta = \max_{0 \leq t \leq T} y(t) \leq C(T)$$

and

$$\int_0^T \int_{S^{2n+1}} |\nabla_\theta(R_\theta + \delta)|_\theta^{\frac{n+1}{2}} dV_\theta dt \leq C(T).$$

By Lemma 2.8, we have

$$\max_{0 \leq t \leq T} \int_{S^{2n+1}} (R_\theta + \delta)^{n+1} dV_{\theta_0} \leq C(T)$$

and

$$\int_0^T \int_{S^{2n+1}} |\nabla_{\theta_0}(R_\theta + \delta)|_{\theta_0}^{\frac{n+1}{2}} dV_{\theta_0} dt \leq C(T).$$

By these estimates and by Lemma 2.3 and Lemma 2.8, we have

$$\begin{aligned}
\int_0^T \left( \int_{S^{2n+1}} (R_\theta + \delta)^{\frac{n+1}{2} \cdot (2 + \frac{2}{n})} dV_\theta \right)^{\frac{n}{n+1}} dt & \leq C(T) \int_0^T \left( \int_{S^{2n+1}} (R_\theta + \delta)^{\frac{n+1}{2} \cdot (2 + \frac{2}{n})} dV_{\theta_0} \right)^{\frac{n}{n+1}} dt \\
& \leq C(T) \int_0^T E((R_\theta + \delta)^{\frac{n+1}{2}}) dt \leq C(T).
\end{aligned}$$

This together with Proposition 2.1 and Lemma 2.4 implies that

$$(2.22) \quad \begin{aligned} & \int_0^T \left( \int_{S^{2n+1}} |\alpha f - R_\theta|^{\frac{(n+1)^2}{n}} dV_\theta \right)^{\frac{n}{n+1}} dt \\ & \leq \int_0^T \left( \int_{S^{2n+1}} (R_\theta + \delta)^{\frac{(n+1)^2}{n}} dV_\theta \right)^{\frac{n}{n+1}} dt + \int_0^T \left( \int_{S^{2n+1}} (\alpha f + \delta)^{\frac{(n+1)^2}{n}} dV_\theta \right)^{\frac{n}{n+1}} dt \leq C(T). \end{aligned}$$

Applying Lemma 2.9 with  $p = \frac{(n+1)^2}{n}$ , we obtain

$$\frac{d}{dt} \log \left( \int_{S^{2n+1}} |\alpha f - R_\theta|^{\frac{(n+1)^2}{n}} dV_\theta \right) \leq C + C \left( \int_{S^{2n+1}} (\alpha f + \delta)^{\frac{(n+1)^2}{n}} dV_\theta \right)^{\frac{n}{n+1}}.$$

Integrating it from 0 to  $T$  and using the estimate (2.22), we can conclude that

$$\int_{S^{2n+1}} |\alpha f - R_\theta|^{\frac{(n+1)^2}{n}} dV_\theta \leq C(T).$$

Now the assertion follows from this, Hölder's inequality and Proposition 2.1.  $\square$

**Lemma 2.12.** *For  $0 < \lambda < \frac{2}{n+1}$  and any fixed  $T > 0$ , there exists a constant  $C = C(T) > 0$  such that*

$$|u(x_1, t_1) - u(x_2, t_2)| \leq C((t_1 - t_2)^{\frac{\lambda}{2}} + d_{S^{2n+1}}(x_1, x_2)^\lambda)$$

for all  $x_1, x_2 \in S^{2n+1}$  and all  $t_1, t_2 \geq 0$  satisfying  $0 < t_1 - t_2 < 1$ . Here  $d_{S^{2n+1}}$  is the Carnot-Carathéodory distance with respect to the contact form  $\theta_0$ .

*Proof.* Choose  $\lambda = 2 - \frac{2n+2}{p}$  with  $n+1 < p < \frac{(n+1)^2}{n}$ . For  $0 \leq t \leq T$  we have

$$(2.23) \quad \begin{aligned} \int_{S^{2n+1}} \left| -\left(2 + \frac{2}{n}\right) \Delta_{\theta_0} u + R_{\theta_0} u \right|^p dV_{\theta_0} &= \int_{S^{2n+1}} |R_{\theta_0} u^{1+\frac{2}{n}}|^p dV_{\theta_0} \\ &\leq C \int_{S^{2n+1}} |R_{\theta_0}|^p dV_{\theta_0} \\ &\leq C \left( \int_{S^{2n+1}} |\alpha f - R_\theta|^p dV_{\theta_0} + \int_{S^{2n+1}} |\alpha f|^p dV_{\theta_0} \right) \\ &\leq C \left( C + (\alpha_2 \max_{S^{2n+1}} f)^p \text{Vol}(S^{2n+1}, \theta_0) \right), \end{aligned}$$

where the first equality follows from (2.6), the first inequality follows from Lemma 2.8, and the final inequality follows from Proposition 2.1, Lemma 2.4 and Lemma 2.11. On the other hand, for  $0 \leq t \leq T$ ,

$$(2.24) \quad \int_{S^{2n+1}} \left| \frac{\partial u}{\partial t} \right|^p dV_{\theta_0} = \left( \frac{n}{2} \right)^p \int_{S^{2n+1}} |(\alpha f - R_\theta) u|^p dV_{\theta_0} \leq C \int_{S^{2n+1}} |\alpha f - R_\theta|^p dV_\theta \leq C,$$

where the first equality follows from (2.5), the first inequality follows from Lemma 2.8, and the last inequality follows from Lemma 2.11.

Then (2.23) implies that

$$|u(x_1, t) - u(x_2, t)| \leq C d_{S^{2n+1}}(x_1, x_2)^\lambda$$

for all  $x_1, x_2 \in S^{2n+1}$  and  $0 \leq t \leq T$ . Now using (2.24), we obtain

$$\begin{aligned}
|u(x, t_1) - u(x, t_2)| &\leq C(t_1 - t_2)^{-(n+1)} \int_{B_{\sqrt{t_1 - t_2}}(x)} |u(x, t_1) - u(x, t_2)| dV_{\theta_0} \\
&\leq C(t_1 - t_2)^{-(n+1)} \int_{B_{\sqrt{t_1 - t_2}}(x)} |u(t_1) - u(t_2)| dV_{\theta_0} + C(t_1 - t_2)^{\frac{\beta}{2}} \\
&\leq C(t_1 - t_2)^{-n} \sup_{t_2 \leq t \leq t_1} \int_{B_{\sqrt{t_1 - t_2}}(x)} \left| \frac{\partial}{\partial t} u(t) \right| dV_{\theta_0} + C(t_1 - t_2)^{\frac{\beta}{2}} \\
&\leq C(t_1 - t_2)^{\frac{\beta}{2}} \sup_{t_2 \leq t \leq t_1} \left( \int_{S^{2n+1}} \left| \frac{\partial}{\partial t} u(t) \right|^p dV_{\theta_0} \right)^{\frac{1}{p}} + C(t_1 - t_2)^{\frac{\beta}{2}} \\
&\leq C(t_1 - t_2)^{\frac{\beta}{2}}
\end{aligned}$$

for all  $x \in S^{2n+1}$  and all  $0 \leq t_1, t_2 \leq T$  satisfying  $0 < t_1 - t_2 < 1$ . This proves the assertion.  $\square$

In view of Lemma 2.12, it is easy to see that all derivatives of  $u(t)$  are uniformly bounded in every finite interval  $[0, T]$ . Indeed, we can apply Theorem 1.1 in [4], which says: let  $X_1, X_2, \dots, X_q$  be a system of real smooth vector fields satisfying Hörmander's condition in a bounded domain  $\Omega$  of  $\mathbb{R}^n$ . Let  $A = \{a_{ij}(t, x)\}_{i,j=1}^q$  be a symmetric, uniformly positive-definite matrix of real functions defined in a domain  $U \subset \mathbb{R} \times \Omega$ . For operator of the form

$$H = \partial_t - \sum_{i,j=1}^q a_{ij}(t, x) X_i X_j - \sum_{i=1}^q b_i(t, x) X_i - c(t, x)$$

we have a priori estimate of Schauder type in parabolic Hörmander Hölder spaces  $C_P^{k,\beta}(U)$ . Namely, for  $a_{ij}, b_i, c \in C_P^{k,\beta}(U)$  and  $U' \Subset U$ , we have

$$(2.25) \quad \|u\|_{C_P^{k+2,\beta}(U')} \leq C\{\|Hu\|_{C_P^{k,\beta}(U)} + \|u\|_{L^\infty(U)}\}.$$

Here, we have (see P.193-194 in [4])

$$\begin{aligned}
C_P^{k,\beta}(U) &= \{u : U \rightarrow \mathbb{R} : \|u\|_{C_P^{k,\beta}(U)} < \infty\}, \\
\|u\|_{C_P^{k,\beta}(U)} &= \sum_{|I|+2h \leq k} \|\partial_t^h X^I u\|_{C_P^\beta(U)}, \\
(2.26) \quad \|u\|_{C_P^\beta(U)} &= |u|_{C_P^\beta(U)} + \|u\|_{L^\infty(U)}, \\
|u|_{C_P^\beta(U)} &= \sup \left\{ \frac{|u(t, x) - u(s, y)|}{d_P((t, x), (s, y))^\beta} : (t, x), (s, y) \in U, (t, x) \neq (s, y) \right\},
\end{aligned}$$

where  $d_P$  is the parabolic Carnot-Carathéodory distance (see P. 189 in [4]) which is given by

$$d_P((t_1, x_1), (t_2, x_2)) = \sqrt{d(x_1, x_2)^2 + |t_1 - t_2|}.$$

Here  $d$  is the Carnot-Carathéodory distance in  $\Omega$ . Moreover, for any multiindex  $I = (i_1, i_2, \dots, i_s)$ , with  $1 \leq i_j \leq q$ ,  $X^I u = X_{i_1} X_{i_2} \dots X_{i_s} u$ .

It follows from Lemma 2.12 that  $u(t, x) \in C_P^{0,\lambda}([0, T] \times S^{2n+1})$ . Therefore, with the estimates (2.25), Lemma 2.8 and 2.15, we can now use the similar standard regularity theory for weakly parabolic equation to show that all higher order derivatives

of  $u(t)$  are uniformly bounded on  $[0, T]$ . This shows the long time existence of the flow (2.3) and (2.5).

### 3. $S_1^2$ AND $L^p$ CONVERGENCE

We first consider  $F_2(t)$ . It follows from (2.18) with  $p = 2$  that

$$\begin{aligned}
 (3.1) \quad \frac{1}{2} \frac{d}{dt} F_2(t) &= \frac{d}{dt} \left( \frac{1}{2} \int_{S^{2n+1}} (\alpha f - R_\theta)^2 dV_\theta \right) \\
 &= \alpha' \int_{S^{2n+1}} f(\alpha f - R_\theta) dV_\theta + \int_{S^{2n+1}} (\alpha f - R_\theta)^2 R_\theta dV_\theta \\
 &\quad - (n+1) \int_{S^{2n+1}} |\nabla_\theta(\alpha f - R_\theta)|_\theta^2 dV_\theta + \frac{n+1}{2} \int_{S^{2n+1}} (\alpha f - R_\theta)^3 dV_\theta.
 \end{aligned}$$

Now if we substitute (2.16) into the first term on the right hand side of (3.1), we obtain

$$\begin{aligned}
 (3.2) \quad \frac{1}{2} \frac{d}{dt} F_2(t) &= -\frac{1}{n+1} \left( \frac{\alpha'}{\alpha} \right)^2 E(u) + \frac{n}{n+1} \frac{\alpha'}{\alpha} \int_{S^{2n+1}} (\alpha f - R_\theta) R_\theta dV_\theta \\
 &\quad + \int_{S^{2n+1}} (\alpha f - R_\theta)^2 R_\theta dV_\theta - (n+1) \int_{S^{2n+1}} |\nabla_\theta(\alpha f - R_\theta)|_\theta^2 dV_\theta \\
 &\quad + \frac{n+1}{2} \int_{S^{2n+1}} (\alpha f - R_\theta)^3 dV_\theta.
 \end{aligned}$$

First we show that the Webster scalar curvature  $R_\theta$  converges to  $\alpha f$  in the  $L^2$  sense.

**Lemma 3.1.** *For a positive smooth solution  $u$  of (2.5), there holds*

$$F_2(t) = \int_{S^{2n+1}} (\alpha f - R_\theta)^2 dV_\theta \rightarrow 0, \text{ as } t \rightarrow \infty.$$

*Proof.* By (2.17), (2.14), Hölder's inequality, Proposition 2.1, Lemma 2.4, we have

$$\begin{aligned}
 (3.3) \quad |\alpha'| &\leq \frac{\alpha}{E(u)} \left[ n \int_{S^{2n+1}} (\alpha f - R_\theta)^2 dV_\theta + \int_{S^{2n+1}} \alpha f |\alpha f - R_\theta| dV_\theta \right] \\
 &\leq \frac{\alpha_2}{R_{\theta_0} \text{Vol}(S^{2n+1}, \theta_0)} \left[ n F_2(t) + \alpha_2 M (\text{Vol}(S^{2n+1}, \theta_0) F_2(t))^{\frac{1}{2}} \right].
 \end{aligned}$$

Thus it follows from (3.1) that

$$\begin{aligned}
 (3.4) \quad \frac{1}{2} \frac{d}{dt} F_2(t) &+ (n+1) \int_{S^{2n+1}} |\nabla_\theta(\alpha f - R_\theta)|_\theta^2 dV_\theta \\
 &= \alpha' \int_{S^{2n+1}} f(\alpha f - R_\theta) dV_\theta + \alpha \int_{S^{2n+1}} f(\alpha f - R_\theta)^2 dV_\theta + \frac{n-1}{2} \int_{S^{2n+1}} (\alpha f - R_\theta)^3 dV_\theta \\
 &\leq C |\alpha'| \left( \int_{S^{2n+1}} (\alpha f - R_\theta)^2 dV_\theta \right)^{\frac{1}{2}} + \alpha \int_{S^{2n+1}} f(\alpha f - R_\theta)^2 dV_\theta - \frac{(n-1)\gamma}{2} \int_{S^{2n+1}} (\alpha f - R_\theta)^2 dV_\theta \\
 &\leq C \int_{S^{2n+1}} (\alpha f - R_\theta)^2 dV_\theta \left[ 1 + \left( \int_{S^{2n+1}} (\alpha f - R_\theta)^2 dV_\theta \right)^{\frac{1}{2}} \right]
 \end{aligned}$$

where we have used Hölder's inequality and Lemma 2.6 in the first inequality, and the last inequality follows from (3.3) and Lemma 2.4.



Hence, if we set  $v(t) = \int_0^{F_2(t)} \frac{ds}{1+s^{1/2}}$ , then

$$\frac{dv(t)}{dt} = \frac{1}{1+\sqrt{F_2(t)}} \frac{d}{dt} F_2(t) \leq C F_2(t).$$

Thus we obtain, for all  $t \geq t_j$

$$v(t) \leq v(t_j) + C \int_{t_j}^{\infty} F_2(t) dt,$$

where  $t_j$  is a sequence defined by (2.13). Note that  $v(t_j) = \int_0^{F_2(t_j)} \frac{ds}{1+s^{1/2}} \leq F(t_j) \rightarrow 0$  as  $j \rightarrow \infty$  by (2.13). Note also that  $\int_{t_j}^{\infty} F_2(t) dt \rightarrow 0$  as  $j \rightarrow \infty$  by (2.12). Therefore,  $v(t) \rightarrow 0$  as  $t \rightarrow \infty$ . On the other hand, by definition of  $v(t)$ , we have

$$(3.5) \quad v(t) = \int_0^{F_2(t)} \frac{ds}{1+s^{1/2}} \geq \frac{F_2(t)}{1+F_2(t)^{\frac{1}{2}}} = F_2(t)^{\frac{1}{2}} - \frac{F_2(t)^{\frac{1}{2}}}{1+F_2(t)^{\frac{1}{2}}},$$

or equivalently,

$$F_2(t)^{\frac{1}{2}} \leq v(t) + \frac{F_2(t)^{\frac{1}{2}}}{1+F_2(t)^{\frac{1}{2}}} \leq v(t) + 1.$$

This implies that  $F_2(t)$  is bounded. By (3.5), we have  $F_2(t) \leq (1+F_2(t)^{\frac{1}{2}})v(t) \leq C v(t) \rightarrow 0$  as  $t \rightarrow \infty$ . This proves the assertion.  $\square$

For  $p \geq 1$ , let

$$G_p(t) = \int_{S^{2n+1}} |\nabla_{\theta}(R_{\theta} - \alpha f)|_{\theta}^p dV_{\theta}.$$

Integrating (3.4) from 0 to  $\infty$  yields

$$(3.6) \quad \int_0^{\infty} G_2(t) dt = \int_0^{\infty} \int_{S^{2n+1}} |\nabla_{\theta}(\alpha f - R_{\theta})|_{\theta}^2 dV_{\theta} dt \leq C + C \int_0^{\infty} F_2(t) dt < \infty$$

by (2.12) and Lemma 3.1. By (3.1), we have

$$\begin{aligned} & \frac{n-1}{2} \int_{S^{2n+1}} (\alpha f - R_{\theta})^3 dV_{\theta} \\ &= \frac{1}{2} \frac{d}{dt} F_2(t) - \alpha \int_{S^{2n+1}} f(\alpha f - R_{\theta})^2 dV_{\theta} + (n+1)G_2(t) - \alpha' \int_{S^{2n+1}} f(\alpha f - R_{\theta}) dV_{\theta} \\ &\leq \frac{1}{2} \frac{d}{dt} F_2(t) + (n+1)G_2(t) + |\alpha'| \left( \max_{S^{2n+1}} f \right) F_1(t) \\ &\leq \frac{1}{2} \frac{d}{dt} F_2(t) + (n+1)G_2(t) \\ &\quad + \frac{2\alpha_2}{n(n+1)\text{Vol}(S^{2n+1}, \theta_0)} \left[ nF_2(t) + \alpha_2 M (\text{Vol}(S^{2n+1}, \theta_0) F_2(t))^{\frac{1}{2}} \right] \left( \max_{S^{2n+1}} f \right) (\text{Vol}(S^{2n+1}, \theta_0) F_2(t))^{\frac{1}{2}} \\ &\leq \frac{1}{2} \frac{d}{dt} F_2(t) + C F_2(t) + (n+1)G_2(t), \end{aligned}$$

where we have used (3.3), Proposition 2.1 and Hölder's inequality in the second inequality, and Lemma 3.1 in the last inequality. Similarly, we can show that

$$\begin{aligned}
& \frac{n-1}{2} \int_{S^{2n+1}} (\alpha f - R_\theta)^3 dV_\theta \\
&= \frac{1}{2} \frac{d}{dt} F_2(t) - \alpha \int_{S^{2n+1}} f(\alpha f - R_\theta)^2 dV_\theta + (n+1)G_2(t) - \alpha' \int_{S^{2n+1}} f(\alpha f - R_\theta) dV_\theta \\
&\geq \frac{1}{2} \frac{d}{dt} F_2(t) - |\alpha| \left( \max_{S^{2n+1}} f \right) F_2(t) - |\alpha'| \left( \max_{S^{2n+1}} f \right) F_1(t) \geq \frac{1}{2} \frac{d}{dt} F_2(t) - CF_2(t).
\end{aligned}$$

Integrating these two inequalities and using (2.12), (3.6) and Lemma 3.1, we conclude that

$$(3.7) \quad \left| \int_0^\infty \int_{S^{2n+1}} (\alpha f - R_\theta)^3 dV_\theta dt \right| < \infty.$$

**Lemma 3.2.** *For any  $p < \infty$ , there holds  $F_p(t) \rightarrow 0$  as  $t \rightarrow \infty$ .*

*Proof.* We divide the proof into two steps.

*Step 1.* We are going to show that there exists  $p_0 > n+1$  and  $v_0 \in (0, 1]$  such that

$$(3.8) \quad \int_0^\infty \left( \int_{S^{2n+1}} |\alpha f - R_\theta|^{p_0} dV_\theta \right)^{v_0} dt < \infty.$$

To do this, we will show that the following estimates are true for all positive integer  $k < \frac{n+1}{2}$ :

$$(3.9) \quad \int_0^\infty \int_{S^{2n+1}} |\alpha f - R_\theta|^{2k} dV_\theta dt < \infty,$$

$$(3.10) \quad \int_{S^{2n+1}} |\alpha f - R_\theta|^{2k} dV_\theta \leq C,$$

$$(3.11) \quad \int_0^\infty \int_{S^{2n+1}} (\alpha f - R_\theta)^{2(k-1)} |\nabla_\theta (\alpha f - R_\theta)|_\theta^2 dV_\theta dt < \infty,$$

$$(3.12) \quad \left| \int_0^\infty \int_{S^{2n+1}} (\alpha f - R_\theta)^{2k+1} dV_\theta dt \right| < \infty,$$

where  $C$  is a positive constant independent of  $t$ .

For  $k = 1$ , (3.9)-(3.12) follow from (2.12), (3.6), (3.7) and Lemma 3.1.

Now suppose that the estimates (3.9)-(3.12) are true for  $k$  with  $k < (n+1)/2$ . By (3.9) and (3.11) and the sharp Folland-Stein inequality (see [20] or Theorem 3.13 in [18]), we obtain

$$(3.13) \quad \int_0^\infty \left( \int_{S^{2n+1}} |\alpha f - R_\theta|^{\left(\frac{2n+2}{n}\right)k} dV_\theta \right)^{\frac{n}{n+1}} dt \leq C.$$

In the following three cases, we intend to show either this procedure terminates by suitable choice of  $p_0 > n+1$  and  $v_0 \in (0, 1]$ , or the estimates (3.9)-(3.12) are true for  $k+1$ .

*Case (a).* If  $k > n/2$ , we can set  $p_0 = k(2n+2)/n > n+1$  and  $v_0 = n/(n+1)$ , then the iteration terminates by (3.13).

Case (b). If  $k < n/2$ , by (2.18) with  $p = 2k + 1$ , we have

$$\begin{aligned}
 (3.14) \quad & \frac{d}{dt} \left( \int_{S^{2n+1}} (\alpha f - R_\theta)^{2k+1} dV_\theta \right) \\
 & + 2k(2k+1)(n+1) \int_{S^{2n+1}} (\alpha f - R_\theta)^{2k-1} |\nabla_\theta(\alpha f - R_\theta)|_\theta^2 dV_\theta \\
 & = (2k+1) \int_{S^{2n+1}} \alpha f (\alpha f - R_\theta)^{2k+1} dV_\theta + (2k+1) \alpha' \int_{S^{2n+1}} f (\alpha f - R_\theta)^{2k} dV_\theta \\
 & + (n-2k) \int_{S^{2n+1}} (\alpha f - R_\theta)^{2k+2} dV_\theta.
 \end{aligned}$$

Note that by Hölder's inequality, Young's inequality, and Lemma 2.4, we have

$$\begin{aligned}
 (3.15) \quad & (2k+1) \left| \int_{S^{2n+1}} \alpha f (\alpha f - R_\theta)^{2k+1} dV_\theta \right| \\
 & \leq (2k+1) \left( \int_{S^{2n+1}} (\alpha f - R_\theta)^{2k+2} dV_\theta \right)^{\frac{1}{2}} \left( \int_{S^{2n+1}} \alpha^2 f^2 (\alpha f - R_\theta)^{2k} dV_\theta \right)^{\frac{1}{2}} \\
 & \leq \frac{n-2k}{2} \int_{S^{2n+1}} (\alpha f - R_\theta)^{2k+2} dV_\theta + C \int_{S^{2n+1}} (\alpha f - R_\theta)^{2k} dV_\theta.
 \end{aligned}$$

By (3.3) and Lemma 3.1, we have

$$(3.16) \quad |\alpha'| \leq C$$

where  $C$  is a constant independent of  $t$ . Combining (3.14), (3.15), (3.16) and Lemma 2.7, we obtain

$$\begin{aligned}
 & \frac{n-2k}{2} \int_{S^{2n+1}} (\alpha f - R_\theta)^{2k+2} dV_\theta \\
 & \leq \frac{d}{dt} \left( \int_{S^{2n+1}} (\alpha f - R_\theta)^{2k+1} dV_\theta \right) + C \int_{S^{2n+1}} (\alpha f - R_\theta)^{2k} dV_\theta \\
 & \quad - 2k(2k+1)(n+1) \gamma \int_{S^{2n+1}} (\alpha f - R_\theta)^{2k-2} |\nabla_\theta(\alpha f - R_\theta)|_\theta^2 dV_\theta.
 \end{aligned}$$

Integrating it from 0 to  $t$ , we get

$$\begin{aligned}
 & \frac{n-2k}{2} \int_0^t \int_{S^{2n+1}} (\alpha f - R_\theta)^{2k+2} dV_\theta dt \\
 & \leq \int_{S^{2n+1}} (\alpha f - R_\theta)^{2k+1}(t) dV_{\theta(t)} - \int_{S^{2n+1}} (\alpha f - R_\theta)^{2k+1}(0) dV_{\theta(0)} \\
 & \quad + C \int_0^t \int_{S^{2n+1}} |\alpha f - R_\theta|^{2k} dV_\theta dt \\
 & \quad - 2k(2k+1)(n+1) \gamma \int_0^t \int_{S^{2n+1}} |\alpha f - R_\theta|^{2(k-1)} |\nabla_\theta(\alpha f - R_\theta)|_\theta^2 dV_\theta dt \\
 & \leq -\gamma \int_{S^{2n+1}} (\alpha f - R_\theta)^{2k}(t) dV_{\theta(t)} + C \leq C
 \end{aligned}$$

where we have used Lemma 2.7 and (3.9)-(3.12) for  $k$ . By letting  $t \rightarrow \infty$ , we have shown that (3.9) holds for  $k+1$ .

Case (b)(i). If  $2k+2 > n+1$ , choose  $p_0 = 2k+2$  and  $v_0 = 1$  in (3.8), then the iteration terminates.

Case (b)(ii). If  $2k + 2 = n + 1$ , by (2.18) with  $p = 2k + \frac{4}{3}$ , we have

$$\begin{aligned}
 & \frac{d}{dt} F_{2k+\frac{4}{3}}(t) + C \int_{S^{2n+1}} |\alpha f - R_\theta|^{2k-\frac{2}{3}} |\nabla_\theta(\alpha f - R_\theta)|_\theta^2 dV_\theta \\
 &= (2k + \frac{4}{3}) \int_{S^{2n+1}} \alpha f |\alpha f - R_\theta|^{2k+\frac{4}{3}} dV_\theta \\
 &+ (2k + \frac{4}{3}) \alpha' \int_{S^{2n+1}} f |\alpha f - R_\theta|^{2k-\frac{2}{3}} (\alpha f - R_\theta) dV_\theta \\
 &+ \frac{2}{3} \int_{S^{2n+1}} |\alpha f - R_\theta|^{2k+\frac{4}{3}} (\alpha f - R_\theta) dV_\theta.
 \end{aligned} \tag{3.17}$$

By Lemma 2.7 and Young's inequality

$$F_{2k+\frac{4}{3}}(t) \leq \frac{1}{3} F_{2k}(t) + \frac{2}{3} F_{2k+2}(t), \tag{3.18}$$

we have

$$\int_{S^{2n+1}} |\alpha f - R_\theta|^{2k+\frac{4}{3}} (\alpha f - R_\theta) dV_\theta \leq -\gamma F_{2k+\frac{4}{3}}(t) \leq -\gamma \left( \frac{1}{3} F_{2k}(t) + \frac{2}{3} F_{2k+2}(t) \right).$$

By (3.16) and Young's inequality  $F_{2k+\frac{1}{3}}(t) \leq \frac{5}{6} F_{2k}(t) + \frac{1}{6} F_{2k+2}(t)$ , we have

$$\left| \alpha' \int_{S^{2n+1}} f |\alpha f - R_\theta|^{2k-\frac{2}{3}} (\alpha f - R_\theta) dV_\theta \right| \leq |\alpha'| \left( \max_{S^{2n+1}} f \right) F_{2k+\frac{1}{3}}(t) \leq C[F_{2k}(t) + F_{2k+2}(t)].$$

Combining all these inequality, we get by (3.17)

$$\begin{aligned}
 & \frac{d}{dt} \left( \int_{S^{2n+1}} (\alpha f - R_\theta)^{2k+\frac{4}{3}} dV_\theta \right) + C \int_{S^{2n+1}} |\alpha f - R_\theta|^{2k-\frac{2}{3}} |\nabla_\theta(\alpha f - R_\theta)|_\theta^2 dV_\theta \\
 & \leq C[F_{2k}(t) + F_{2k+2}(t)].
 \end{aligned} \tag{3.19}$$

Due to (3.9) for  $k$  and  $k + 1$ , integrating (3.19) from 0 to  $\infty$  with respect to  $t$  yields

$$\int_0^\infty \int_{S^{2n+1}} |\alpha f - R_\theta|^{2k-\frac{2}{3}} |\nabla_\theta(\alpha f - R_\theta)|_\theta^2 dV_\theta dt \leq C \quad \text{and} \quad F_{2k+\frac{4}{3}}(t) \leq C. \tag{3.20}$$

Also, due to (3.9) for  $k$  and  $k + 1$ , integrating (3.18) from 0 to  $\infty$  with respect to  $t$  yields

$$\int_0^\infty \int_{S^{2n+1}} |\alpha f - R_\theta|^{2k+\frac{4}{3}} dV_\theta dt \leq C. \tag{3.21}$$

Combining (3.20) and (3.21) and using the sharp Folland-Stein inequality again, we obtain

$$\int_0^\infty \left( \int_{S^{2n+1}} |\alpha f - R_\theta|^{(k+\frac{2}{3})(2+\frac{2}{n})} dV_\theta \right)^{\frac{n}{n+1}} dt \leq C.$$

Now we can choose  $p_0 = (k + \frac{2}{3})(2 + \frac{2}{n}) > \frac{n+1}{2}$  and  $v_0 = \frac{n}{n+1}$  and then the iteration terminates.

Case (b)(iii). If  $2k+2 < n+1$ , by (2.18) again with  $p = 2k+2$ , we have

$$\begin{aligned}
 & \frac{d}{dt} F_{2k+2}(t) + C \int_{S^{2n+1}} |\alpha f - R_\theta|^{2k} |\nabla_\theta(\alpha f - R_\theta)|_\theta^2 dV_\theta \\
 &= (2k+2) \int_{S^{2n+1}} \alpha f |\alpha f - R_\theta|^{2k+2} dV_\theta \\
 (3.22) \quad &+ (2k+2)\alpha' \int_{S^{2n+1}} f |\alpha f - R_\theta|^{2k} (\alpha f - R_\theta) dV_\theta \\
 &+ (n-2k-1) \int_{S^{2n+1}} |\alpha f - R_\theta|^{2k+2} (\alpha f - R_\theta) dV_\theta \\
 &\leq C F_{2k+2}(t) + C F_2(t) - \gamma(n-2k-1) F_{2k+2}(t)
 \end{aligned}$$

where we have used (3.16), Lemma 2.7, and the fact that

$$|\alpha f - R_\theta|^{2k+1} \leq \begin{cases} |\alpha f - R_\theta|^{2k+2}, & \text{if } |\alpha f - R_\theta| \geq 1; \\ |\alpha f - R_\theta|^2, & \text{if } |\alpha f - R_\theta| < 1. \end{cases}$$

By (2.12) and (3.9) for  $k+1$ , integrating (3.22) from 0 to  $\infty$  yields

$$(3.23) \quad \int_0^\infty \int_{S^{2n+1}} |\alpha f - R_\theta|^{2k} |\nabla_\theta(\alpha f - R_\theta)|_\theta^2 dV_\theta dt \leq C \quad \text{and} \quad F_{2k+2}(t) \leq C.$$

This implies that (3.10) and (3.11) are true for  $k+1$ . By (3.22) again, we have

$$\begin{aligned}
 & (n-2k-1) \int_{S^{2n+1}} |\alpha f - R_\theta|^{2k+2} (\alpha f - R_\theta) dV_\theta \\
 &= \frac{d}{dt} F_{2k+2}(t) + C \int_{S^{2n+1}} |\alpha f - R_\theta|^{2k} |\nabla_\theta(\alpha f - R_\theta)|_\theta^2 dV_\theta \\
 &\quad - (2k+2) \int_{S^{2n+1}} \alpha f |\alpha f - R_\theta|^{2k+2} dV_\theta - (2k+2)\alpha' \int_{S^{2n+1}} f |\alpha f - R_\theta|^{2k} (\alpha f - R_\theta) dV_\theta \\
 &\leq \frac{d}{dt} F_{2k+2}(t) + C \int_{S^{2n+1}} |\alpha f - R_\theta|^{2k} |\nabla_\theta(\alpha f - R_\theta)|_\theta^2 dV_\theta + C F_{2k+2}(t) + C F_2(t).
 \end{aligned}$$

Integrating it from 0 to  $\infty$ , we obtain by (2.12), (3.23) and by (3.9) for  $k+1$

$$\left| \int_0^\infty \int_{S^{2n+1}} (\alpha f - R_\theta)^{2k+3} dV_\theta dt \right| < \infty,$$

which implies that (3.12) is true for  $k+1$ . Thus, the iteration procedure can be continued.

Case (c). If  $k = \frac{n}{2}$ , we consider (2.18) with  $p = 2k + \frac{1}{3}$ . It yields

$$\begin{aligned}
 (3.24) \quad & \frac{d}{dt} F_{2k+\frac{1}{3}}(t) + C \int_{S^{2n+1}} |\alpha f - R_\theta|^{2k-\frac{5}{3}} |\nabla_\theta(\alpha f - R_\theta)|_\theta^2 dV_\theta \\
 &\leq \frac{6k+1}{3} \int_{S^{2n+1}} \alpha f |\alpha f - R_\theta|^{\frac{6k+1}{3}} dV_\theta + \frac{6k+1}{3} \alpha' \int_{S^{2n+1}} f |\alpha f - R_\theta|^{\frac{6k-5}{3}} (\alpha f - R_\theta) dV_\theta \\
 &\quad + \frac{2}{3} \int_{S^{2n+1}} |\alpha f - R_\theta|^{\frac{6k+1}{3}} (\alpha f - R_\theta) dV_\theta.
 \end{aligned}$$

By Lemma 2.7, we have

$$(3.25) \quad \int_{S^{2n+1}} |\alpha f - R_\theta|^{\frac{6k+1}{3}} (\alpha f - R_\theta) dV_\theta \leq -\gamma F_{\frac{6k+1}{3}}(t).$$

By Lemma 2.4, we have

$$\int_{S^{2n+1}} \alpha f |\alpha f - R_\theta|^{\frac{6k+1}{3}} dV_\theta \leq C F_{\frac{6k+1}{3}}(t).$$

By Hölder's inequality and Young's inequality, we get

$$\begin{aligned} (3.26) \quad F_{\frac{6k+1}{3}}(t) &\leq F_{k(\frac{2n+2}{n})}(t)^{\frac{1}{3}} F_{2k}(t)^{\frac{2}{3}} \\ &\leq \frac{n+1}{3n} F_{k(\frac{2n+2}{n})}(t)^{\frac{n}{n+1}} + \frac{2n-1}{3n} F_{2k}(t)^{\frac{2n}{2n-1}} \\ &\leq C F_{k(\frac{2n+2}{n})}(t)^{\frac{n}{n+1}} + C F_{2k}(t) \end{aligned}$$

where we have used (3.10) and the fact that  $k = \frac{n}{2}$ . On the other hand, by (3.3) and Lemma 3.1, we can estimate

$$\begin{aligned} \left| \alpha' \int_{S^{2n+1}} f |\alpha f - R_\theta|^{\frac{6k-5}{3}} (\alpha f - R_\theta) dV_\theta \right| &\leq C |\alpha'| F_{\frac{6k-2}{3}}(t) \\ &\leq C (F_2(t) + F_2(t)^{\frac{1}{2}}) F_{\frac{6k-2}{3}}(t) \\ &\leq C (F_2(t) + F_2(t)^{\frac{1}{2}}) (F_{2k}(t) + F_1(t)) \\ &\leq C (F_2(t) + F_2(t)^{\frac{1}{2}}) (F_{2k}(t) + F_2(t)^{\frac{1}{2}}) \\ &\leq C F_{2k}(t) + C F_2(t), \end{aligned}$$

where we have used the inequality

$$|\alpha f - R_\theta|^{\frac{6k-2}{3}} \leq \begin{cases} |\alpha f - R_\theta|^{2k}, & \text{if } |\alpha f - R_\theta| \geq 1; \\ |\alpha f - R_\theta|, & \text{if } |\alpha f - R_\theta| < 1. \end{cases}$$

Combining the above estimates, the right hand side of (3.24) can be bounded by

$$\begin{aligned} &\frac{d}{dt} F_{2k+\frac{1}{3}}(t) + C \int_{S^{2n+1}} |\alpha f - R_\theta|^{2k-\frac{5}{3}} |\nabla_\theta(\alpha f - R_\theta)|_\theta^2 dV_\theta \\ &\leq C F_{k(\frac{2n+2}{n})}(t)^{\frac{n}{n+1}} + C F_{2k}(t) + C F_2(t). \end{aligned}$$

Integrating it from 0 to  $\infty$ , we obtain

$$(3.27) \quad \int_0^\infty \int_{S^{2n+1}} |\alpha f - R_\theta|^{2k-\frac{5}{3}} |\nabla_\theta(\alpha f - R_\theta)|_\theta^2 dV_\theta dt \leq C \text{ and } F_{2k+\frac{1}{3}}(t) \leq C$$

by (2.12), (3.9) and (3.13). On the other hand, integrating (3.26) from 0 to  $\infty$ , we obtain

$$(3.28) \quad \int_0^\infty \int_{S^{2n+1}} |\alpha f - R_\theta|^{2k+\frac{1}{3}} dV_\theta dt \leq C$$

by (3.9) and (3.13). Using (3.27), (3.28) and the sharp Folland-Stein inequality again, we obtain

$$\int_0^\infty \left( \int_{S^{2n+1}} |\alpha f - R_\theta|^{(k+\frac{1}{6})(2+\frac{2}{n})} dV_\theta \right)^{\frac{n}{n+1}} dt < \infty.$$

Hence we can choose  $p_0 = (k + \frac{1}{6})(2 + \frac{2}{n}) = n + 1 + \frac{n+1}{3n}$  since  $k = \frac{n}{2}$  and  $v_0 = \frac{n}{n+1}$  in (3.8), and the iteration terminates.

In sum, after finitely many steps, we obtain (3.8) with some suitable  $p_0 > n + 1$  and  $v_0 \in (0, 1]$ .

*Step 2.* We will show that for any  $p > n + 1$ , there holds  $\lim_{t \rightarrow \infty} F_p(t) = 0$ . Set  $p_k = p_0(\frac{n+1}{n})^k$ ,  $k \in \mathbb{N}$ , where  $p_0, v_0$  are given in Step 1. Next we argue by induction on  $k$ .

Assume for  $p = p_k$  and some  $v_k \in (0, 1]$ , there holds

$$(3.29) \quad \int_0^\infty F_{p_k}(t)^{v_k} dt < \infty.$$

We are going to show that  $\lim_{t \rightarrow \infty} F_{p_k}(t) = 0$  and to establish (3.29) with  $p_{k+1}$  and  $v_k = \frac{n}{n+1}$ . Using (2.19) for  $p = p_k > n + 1$ , we have

$$(3.30) \quad \frac{d}{dt} F_{p_k}(t) + F_{p_{k+1}}(t)^{v_{k+1}} \leq C F_{p_k}(t)^{v_k} \left( F_{p_k}(t)^{1-v_k} + F_{p_k}(t)^{\frac{p_k-n}{p_k-n-1}-v_k} \right).$$

From (3.29), there exists a sequence  $t_j$  with  $t_j \rightarrow \infty$  as  $j \rightarrow \infty$  such that

$$(3.31) \quad F_{p_k}(t_j) \rightarrow 0 \quad \text{as } j \rightarrow \infty.$$

For brevity, let  $F(t) = F_{p_k}(t)$  and  $v = v_k$  with  $\int_0^\infty F(t)^v dt < \infty$  by (3.29). Then (3.30) implies that

$$(3.32) \quad \frac{d}{dt} F(t) \leq C F(t)^v (F(t)^{\beta_1} + F(t)^{\beta_2})$$

where  $0 \leq \beta_1 = 1 - v_k < 1$  and  $\beta_2 = \frac{p_k - n}{p_k - n - 1} - v_k = \beta_1 + \frac{1}{p_k - n - 1} > \beta_1 \geq 0$ . Define

$$H(t) = \int_0^{F(t)} \frac{ds}{s^{\beta_1} + s^{\beta_2}}.$$

By (3.31), we have

$$(3.33) \quad H(t_j) \leq \int_0^{F(t_j)} \frac{ds}{s^{\beta_1}} = \frac{1}{1 - \beta_1} F(t_j)^{1-\beta_1} \rightarrow 0 \quad \text{as } j \rightarrow \infty.$$

By (3.32), we also have

$$\frac{d}{dt} H(t) = \frac{1}{F(t)^{\beta_1} + F(t)^{\beta_2}} \cdot \frac{d}{dt} F(t) \leq C F(t)^v.$$

Integrating it from  $t_j$  to  $t$  yields

$$H(t) \leq H(t_j) + C \int_{t_j}^\infty F(t)^v dt,$$

where the right hand side tends to 0 as  $j \rightarrow \infty$  by (3.29) and (3.33). This implies that

$$(3.34) \quad \lim_{t \rightarrow \infty} H(t) = 0.$$

We claim that there exists a positive constant  $C_0$  such that  $F(t) \leq C_0$  for all  $t$ . Otherwise, there would exist a sequence  $t_k$  with  $t_k \rightarrow \infty$  as  $k \rightarrow \infty$ , such that  $F(t_k) > 1$  for all  $k \in \mathbb{N}$ , which would imply

$$H(t_k) = \int_0^{F(t_k)} \frac{ds}{s^{\beta_1} + s^{\beta_2}} \geq \int_0^1 \frac{ds}{s^{\beta_1} + s^{\beta_2}} > 0, \quad \text{for all } k \in \mathbb{N},$$

which contradicts (3.34). Hence, by definition of  $H(t)$ , we have

$$H(t) \geq \frac{F(t)}{F(t)^{\beta_1} + F(t)^{\beta_2}} \geq \frac{F(t)}{C_0^{\beta_1} + C_0^{\beta_2}} = CF(t).$$

Combining this with (3.34), we have

$$(3.35) \quad \lim_{t \rightarrow \infty} F(t) = 0.$$

By (3.30) and (3.35), we have  $\frac{d}{dt}F(t) + F_{p_{k+1}}(t)^{v_{k+1}} \leq CF(t)^v$ . Integrating it from 0 to  $\infty$  and using (3.29) and (3.35), we obtain

$$\int_0^\infty F_{p_{k+1}}(t)^{v_{k+1}} dt < \infty.$$

Thus, the induction step is complete.  $\square$

**Lemma 3.3.** *There holds  $G_2(t) \rightarrow 0$  as  $t \rightarrow \infty$ .*

*Proof.* By (3.6), there exists a sequence  $t_j$  with  $t_j \rightarrow \infty$  as  $j \rightarrow \infty$  such that

$$(3.36) \quad G_2(t_j) \rightarrow 0 \text{ as } j \rightarrow \infty.$$

Since  $\theta = u^{\frac{2}{n}}\theta_0$ , we have (see (2.4) in [28] for example)

$$\int_{S^{2n+1}} \langle \nabla_\theta \psi_1, \nabla_\theta \psi_2 \rangle_\theta dV_\theta = \int_{S^{2n+1}} u^2 \langle \nabla_{\theta_0} \psi_1, \nabla_{\theta_0} \psi_2 \rangle_{\theta_0} dV_{\theta_0},$$

for any function  $\psi_1$  and  $\psi_2$ , which together with (2.5) and Lemma 2.5 implies that

$$\begin{aligned} \frac{d}{dt}G_2(t) &= 2 \int_{S^{2n+1}} |\nabla_{\theta_0}(\alpha f - R_\theta)|_{\theta_0}^2 u \frac{\partial u}{\partial t} dV_{\theta_0} + 2 \int_{S^{2n+1}} \langle \nabla_\theta(\alpha f - R_\theta), \nabla_\theta(\alpha' f - \frac{\partial R_\theta}{\partial t}) \rangle_\theta dV_\theta \\ &= n \int_{S^{2n+1}} u^2 |\nabla_{\theta_0}(\alpha f - R_\theta)|_{\theta_0}^2 (\alpha f - R_\theta) dV_{\theta_0} + 2\alpha' \int_{S^{2n+1}} \langle \nabla_\theta(\alpha f - R_\theta), \nabla_\theta f \rangle_\theta dV_\theta \\ &\quad - 2 \int_{S^{2n+1}} \Delta_\theta(\alpha f - R_\theta) \left[ (n+1)\Delta_\theta(\alpha f - R_\theta) + (\alpha f - R_\theta)R_\theta \right] dV_\theta \\ &:= I + II + III. \end{aligned}$$

Note that by Lemma 2.7 we have

$$I \leq -n\gamma \int_{S^{2n+1}} u^2 |\nabla_{\theta_0}(\alpha f - R_\theta)|_{\theta_0}^2 dV_{\theta_0} = -n\gamma G_2(t).$$

Note also that  $|\alpha'| \leq CF_2(t) + CF_2(t)^{\frac{1}{2}} \leq CF_2(t)^{\frac{1}{2}}$  by (3.3) and the fact that  $F_2(t) \leq C$ . This together with Hölder's inequality and Young's inequality implies that

$$\begin{aligned} |II| &\leq 2 \frac{|\alpha'|}{2} \left( \int_{S^{2n+1}} |\nabla_\theta(\alpha f - R_\theta)|_\theta^2 dV_\theta \right)^{\frac{1}{2}} \left( \int_{S^{2n+1}} |\nabla_\theta f|_\theta^2 dV_\theta \right)^{\frac{1}{2}} \\ &\leq CF_2(t)^{\frac{1}{2}} G_2(t)^{\frac{1}{2}} \leq CF_2(t) + CG_2(t), \end{aligned}$$

since

$$\int_{S^{2n+1}} |\nabla_\theta f|_\theta^2 dV_\theta = \int_{S^{2n+1}} |\nabla_{\theta_0} f|_{\theta_0}^2 u^2 dV_{\theta_0} \leq C \left( \int_{S^{2n+1}} u^{\frac{2n+2}{n}} dV_{\theta_0} \right)^{\frac{n}{n+1}} \leq C$$



by Proposition 2.1. On the other hand, for any  $\epsilon > 0$ , by Lemma 2.4 and Young's inequality, we have

$$\begin{aligned}
& \frac{III}{2} + (n+1) \int_{S^{2n+1}} |\Delta_\theta(\alpha f - R_\theta)|^2 dV_\theta \\
&= - \int_{S^{2n+1}} (\alpha f - R_\theta) R_\theta \Delta_\theta(\alpha f - R_\theta) dV_\theta \\
&= \int_{S^{2n+1}} (\alpha f - R_\theta)^2 \Delta_\theta(\alpha f - R_\theta) dV_\theta - \alpha \int_{S^{2n+1}} f(\alpha f - R_\theta) \Delta_\theta(\alpha f - R_\theta) dV_\theta \\
&\leq \epsilon \int_{S^{2n+1}} |\Delta_\theta(\alpha f - R_\theta)|^2 dV_\theta + C \int_{S^{2n+1}} (\alpha f - R_\theta)^4 dV_\theta + C \int_{S^{2n+1}} (\alpha f - R_\theta)^2 dV_\theta.
\end{aligned}$$

Hence, if we choose  $\epsilon = n+1$  and by Hölder's inequality, we get

$$\begin{aligned}
III &\leq C[F_4(t) + F_2(t)] \leq C[F_{2n+2}(t)^{\frac{1}{n+1}} F_{\frac{2n+2}{n}}(t)^{\frac{n}{n+1}} + F_2(t)] \\
&\leq C[F_{\frac{2n+2}{n}}(t)^{\frac{n}{n+1}} + F_2(t)]
\end{aligned}$$

since  $F_{2n+2}(t) \leq C$  by Lemma 3.2. Combining all the estimates above, we obtain

$$\frac{d}{dt} G_2(t) \leq C[G_2(t) + F_2(t) + F_{\frac{2n+2}{n}}(t)^{\frac{n}{n+1}}].$$

Integrating it from  $t_j$  to  $t$ , we get

$$G_2(t) \leq G_2(t_j) + C \int_{t_j}^t [G_2(t) + F_2(t) + F_{\frac{2n+2}{n}}(t)^{\frac{n}{n+1}}] dt.$$

Note that the right hand side tends to 0 when  $j \rightarrow \infty$ , thanks to (2.12), (3.6), (3.13) and (3.36). This proves the assertion.  $\square$

#### 4. BLOW-UP ANALYSIS

The Riemannian version of the following theorem was proved by Schwetlick and Struwe (see Theorem A.1 in [40]).

**Theorem 4.1.** *Let  $u \in C^\infty(M)$  be a solution of*

$$(4.1) \quad -\left(2 + \frac{2}{n}\right) \Delta_{\theta_0} u + R_{\theta_0} u = Pu \quad \text{on } S^{2n+1}.$$

(i) *For any  $\sigma < Y(S^{2n+1}, \theta_0)$ , there exists constants  $q_0 > 2 + \frac{2}{n}$  and  $r_0 > 0$  such that whenever for some  $r < r_0$  and some  $x_0 \in M$  there holds  $\|P\|_{L^{n+1}(B_{2r}(x_0))} \leq \sigma$ , then*

$$\|u\|_{L^{q_0}(B_r(x_0))} < C \|u\|_{L^{2+\frac{2}{n}}(B_{2r}(x_0))}$$

*for some constant  $C$  independent of  $x_0$ .*

(ii) *For any  $q > 2 + \frac{2}{n}$ , and any  $r > 0$ , there exists a constant  $C = C(q, r)$  such that*

$$\|u\|_{L^q(B_{3r}(x_0))} < C \|u\|_{L^{2+\frac{2}{n}}(B_{4r}(x_0))}$$

*whenever there holds  $\|P\|_{L^{n+1}(B_{4r}(x_0))} < \frac{2n+2}{nq} Y(S^{2n+1}, \theta_0)$ .*

*Proof.* (i) For suitable  $p \geq 1$  and  $\eta \in C_0^1(B_{2r}(x_0))$  with

$$(4.2) \quad \eta = 1 \text{ in } B_r(x_0), \quad \eta = 0 \text{ in } S^{2n+1} \setminus B_{2r}(x_0), \quad |\nabla_{\theta_0} \eta|_{\theta_0} < C/r,$$

we let  $v = u^{2p-1}\eta^2$ . Multiplying (4.1) by  $v$  and integrating it over  $S^{2n+1}$ , we have

$$(4.3) \quad \int_{S^{2n+1}} \left( \left(2 + \frac{2}{n}\right) \langle \nabla_{\theta_0} u, \nabla_{\theta_0} v \rangle_{\theta_0} + R_{\theta_0} uv \right) dV_{\theta_0} = \int_{S^{2n+1}} Puv dV_{\theta_0}.$$

Also, let  $w = u^p \eta$ . Then  $w^2 = u^{2p} \eta^2 = uv$  and

$$(4.4) \quad \begin{aligned} |\nabla_{\theta_0} w|_{\theta_0}^2 &= p^2 u^{2p-2} |\nabla_{\theta_0} u|_{\theta_0}^2 \eta^2 + 2p u^{2p-1} \eta \langle \nabla_{\theta_0} u, \nabla_{\theta_0} \eta \rangle_{\theta_0} + u^{2p} |\nabla_{\theta_0} \eta|_{\theta_0}^2 \\ &\leq \left(p^2 + \frac{p-1}{2}\right) u^{2p-2} |\nabla_{\theta_0} u|_{\theta_0}^2 \eta^2 + (p+1) u^{2p-1} \eta \langle \nabla_{\theta_0} u, \nabla_{\theta_0} \eta \rangle_{\theta_0} + \left(1 + \frac{p-1}{2}\right) u^{2p} |\nabla_{\theta_0} \eta|_{\theta_0}^2 \\ &= \frac{1+p}{2} \left( \langle \nabla_{\theta_0} u, \nabla_{\theta_0} v \rangle_{\theta_0} + u^{2p} |\nabla_{\theta_0} \eta|_{\theta_0}^2 \right). \end{aligned}$$

Thus,

$$(4.5) \quad \begin{aligned} &\frac{2}{1+p} \int_{S^{2n+1}} \left( \left(2 + \frac{2}{n}\right) |\nabla_{\theta_0} w|_{\theta_0}^2 + R_{\theta_0} w^2 \right) dV_{\theta_0} - \left(2 + \frac{2}{n}\right) \int_{S^{2n+1}} u^{2p} |\nabla_{\theta_0} \eta|_{\theta_0}^2 dV_{\theta_0} \\ &\leq \int_{S^{2n+1}} \left(2 + \frac{2}{n}\right) \langle \nabla_{\theta_0} u, \nabla_{\theta_0} v \rangle_{\theta_0} dV_{\theta_0} + \frac{2}{1+p} \int_{S^{2n+1}} R_{\theta_0} w^2 dV_{\theta_0} \\ &\leq \int_{S^{2n+1}} \left( \left(2 + \frac{2}{n}\right) \langle \nabla_{\theta_0} u, \nabla_{\theta_0} v \rangle_{\theta_0} dV_{\theta_0} + R_{\theta_0} w^2 \right) dV_{\theta_0} \\ &= \int_{B_{2r}(x_0)} Pw^2 dV_{\theta_0} \leq \|P\|_{L^{n+1}(B_{2r}(x_0))} \|w\|_{L^{2+\frac{2}{n}}(B_{2r}(x_0))}^2 \leq \sigma \|w\|_{L^{2+\frac{2}{n}}(B_{2r}(x_0))}^2, \end{aligned}$$

where the first inequality follows from (4.4), and the first equality follows from (4.3) and the fact that support of  $w$  lies in  $B_{2r}(x_0)$ , and the second last inequality follows from Hölder's inequality, and the last inequality follows from the assumption that  $\|P\|_{L^{n+1}(B_{2r}(x_0))} \leq \sigma$ .

On the other hand, by Lemma 2.3 and the fact that support of  $w$  lies in  $B_{2r}(x_0)$ , we have

$$\int_{S^{2n+1}} \left( \left(2 + \frac{2}{n}\right) |\nabla_{\theta_0} w|_{\theta_0}^2 + R_{\theta_0} w^2 \right) dV_{\theta_0} \geq Y(S^{2n+1}, \theta_0) \|w\|_{L^{2+\frac{2}{n}}(B_{2r}(x_0))}^2.$$

Combining this with (4.5), we have

$$(4.6) \quad \left( \frac{2}{1+p} Y(S^{2n+1}, \theta_0) - \sigma \right) \|w\|_{L^{2+\frac{2}{n}}(B_{2r}(x_0))}^2 \leq \left(2 + \frac{2}{n}\right) \int_{S^{2n+1}} u^{2p} |\nabla_{\theta_0} \eta|_{\theta_0}^2 dV_{\theta_0}.$$

Let  $\sigma = aY(S^{2n+1}, \theta_0)$ , then  $a < 1$  by assumption. For  $1 < p < \min \left\{ \frac{3-a}{1+a}, 2 + \frac{2}{n} \right\}$  and  $r < r_0 = r_0(a)$ , we have

$$\begin{aligned} \frac{(1-a)Y(S^{2n+1}, \theta_0)}{4(2 + \frac{2}{n})} \|u\|_{L^{(2+\frac{2}{n})p}(B_r(x_0))}^{2p} &= \frac{(1-a)Y(S^{2n+1}, \theta_0)}{4(2 + \frac{2}{n})} \left( \int_{B_r(x_0)} w^{\frac{2n+2}{n}} dV_{\theta_0} \right)^{\frac{n}{n+1}} \\ &\leq \frac{(1-a)Y(S^{2n+1}, \theta_0)}{4(2 + \frac{2}{n})} \|w\|_{L^{\frac{2n+2}{n}}(B_{2r}(x_0))}^2 \\ &\leq \int_{S^{2n+1}} u^{2p} |\nabla_{\theta_0} \eta|_{\theta_0}^2 dV_{\theta_0} \leq Cr^{-2} \|u\|_{L^{\frac{2n+2}{n}}(B_{2r}(x_0))}^{2p} \end{aligned}$$

where the second last inequality follows from (4.6), and the last inequality follows from (4.2) and Hölder's inequality. This proves the assertion.

(ii) For  $1 < p \leq \frac{q}{2 + \frac{2}{n}}$  and  $\eta \in C^1(B_{4r}(x_0))$ , we define the test function  $v = u^{2p-1}\eta^2$ . Then by the same argument as (i) we can derive

$$(4.7) \quad \frac{Y(S^{2n+1}, \theta_0)}{2 + \frac{2}{n}} \left( \frac{2}{1+p} - \frac{2 + \frac{2}{n}}{q} \right) \|w\|_{L^{2+\frac{2}{n}}(S^{2n+1})}^2 \leq \int_{S^{2n+1}} u^{2p} |\nabla_{\theta_0} \eta|_{\theta_0}^2 dV_{\theta_0}$$

where  $w = u^p \eta$ , since  $\|P\|_{L^{n+1}(B_{4r}(x_0))} < \frac{2n+2}{nq} Y(S^{2n+1}, \theta_0)$  by assumption.

Choose  $p_1 = \left(1, \frac{n+1}{n}\right]$  maximal such that  $q = 2p_1 \left(\frac{n+1}{n}\right)^m$  for some  $m \in \mathbb{N}$ .

For  $i = 1, \dots, m+1$ , let  $p_i = p_1 \left(\frac{n+1}{n}\right)^{i-1}$ ,  $r_i = r(3 + 2^{1-i})$ , and

$$\eta_i = 1 \text{ in } B_{r_{i+1}}(x_0), \quad \eta_i = 0 \text{ in } S^{2n+1} \setminus B_{r_i}(x_0), \quad |\nabla_{\theta_0} \eta_i|_{\theta_0} < C2^i/r,$$

we can apply (4.7) to get

$$\|u\|_{L^{2p_{i+1}}(B_{r_{i+1}}(x_0))} \leq \left( \frac{C2^{2i}(q - \frac{2n+2}{n})}{r^2} \right)^{\frac{1}{2p_i}} \|u\|_{L^{2p_i}(B_{r_i}(x_0))},$$

which can be iterated to obtain the result.  $\square$

Now we can apply the previous theorem to prove the following concentration-compactness result, which is the CR version of the results of Schwetlick and Struwe (see Theorem 3.1 in [40]).

**Theorem 4.2.** *Let  $\theta_k = u_k^{\frac{2}{n}} \theta_0$ , where  $0 < u_k \in C^\infty(S^{2n+1})$  and  $k \in \mathbb{Z}^+$ , be a family of contact form with  $\text{Vol}(S^{2n+1}, \theta_k) = \text{Vol}(S^{2n+1}, \theta_0)$ . If there exists  $p_1 > n+1$  and a constant  $C_0$  such that*

$$(4.8) \quad \bar{R}_{\theta_k} = \frac{\int_{S^{2n+1}} R_{\theta_k} dV_{\theta_k}}{\int_{S^{2n+1}} dV_{\theta_k}} \leq C_0 \text{ and } \int_{S^{2n+1}} |R_{\theta_k} - \bar{R}_{\theta_k}|^{p_1} dV_{\theta_k} \leq C_0$$

for all  $k$ , then either

- (i) the sequence  $\{u_k\}$  is uniformly bounded in  $S_2^p(S^{2n+1}, \theta_0)$  for all  $p < p_1$ , or
- (ii) there exists a subsequence  $\{u_k\}$  (relabelled) and finitely many points  $x_1, \dots, x_L \in S^{2n+1}$  such that for any  $r > 0$  and any  $i \in \{1, \dots, L\}$  there holds

$$(4.9) \quad \liminf_{k \rightarrow \infty} \left( \int_{B_r(x_i)} |R_{\theta_k}|^{n+1} dV_{\theta_k} \right)^{\frac{1}{n+1}} \geq Y(S^{2n+1}, \theta_0).$$

Moreover, the sequence  $\{u_k\}$  is bounded in  $S_2^p(S^{2n+1}, \theta_0)$  on any compact subset of  $(S^{2n+1} \setminus \{x_1, \dots, x_L\}, \theta_0)$ .

*Proof.* Fix a point  $x_0 \in S^{2n+1}$  and assume that for some  $r > 0$  there holds

$$(4.10) \quad \sup_k \left( \int_{B_r(x_0)} |R_{\theta_k}|^{n+1} dV_{\theta_k} \right)^{\frac{1}{n+1}} \leq \sigma < Y(S^{2n+1}, \theta_0).$$

For  $\theta_k = u_k^{\frac{2}{n}} \theta_0$ , we have

$$-(2 + \frac{2}{n}) \Delta_{\theta_0} u_k + R_{\theta_0} u_k = R_{\theta_k} u_k^{1+\frac{2}{n}} = P_k u_k,$$

where  $P_k = R_{\theta_k} u_k^{\frac{2}{n}} \in L^{n+1}(B_{2r}(x_0))$  because

$$\|P_k\|_{L^{n+1}(B_{2r}(x_0))} = \left( \int_{B_{2r}(x_0)} |R_{\theta_k}|^{n+1} u_k^{2+\frac{2}{n}} dV_{\theta_0} \right)^{\frac{1}{n+1}} \leq \sigma < Y(S^{2n+1}, \theta_0)$$

by (4.10). Clearly we can assume that  $r \leq r_0(\sigma)$  as determined in Theorem 4.1(i).

Therefore, Theorem 4.1(i) and the assumption that  $\|u_k\|_{L^{2+\frac{2}{n}}(S^{2n+1})}^{2+\frac{2}{n}} = \text{Vol}(S^{2n+1}, \theta_k) =$

$\text{Vol}(S^{2n+1}, \theta_0)$  imply that  $u_k$  is bounded in  $L^{q_0}(B_r(x_0))$  for some  $q_0 > 2 + \frac{2}{n}$ . In particular,

$$(4.11) \quad \text{Vol}(B_r(x_0), \theta_k) \leq \|u_k\|_{L^{q_0}(B_r(x_0))}^{2+\frac{2}{n}} \text{Vol}(B_r(x_0), \theta_0)^{1-\frac{1}{q_0}} \rightarrow 0 \quad \text{as } r \rightarrow 0$$

uniformly in  $k \in \mathbb{Z}^+$ . Then

$$\begin{aligned} & \|P_k\|_{L^{n+1}(B_{4r}(x_0))} \\ &= \left( \int_{B_{4r}(x_0)} |R_{\theta_k}|^{n+1} u_k^{2+\frac{2}{n}} dV_{\theta_0} \right)^{\frac{1}{n+1}} = \left( \int_{B_{4r}(x_0)} |R_{\theta_k}|^{n+1} dV_{\theta_k} \right)^{\frac{1}{n+1}} \\ &\leq \left( \int_{B_{4r}(x_0)} |R_{\theta_k} - \bar{R}_{\theta_k}|^{n+1} dV_{\theta_k} \right)^{\frac{1}{n+1}} + \bar{R}_{\theta_k} \left( \int_{B_{4r}(x_0)} dV_{\theta_k} \right)^{\frac{1}{n+1}} \\ &\leq \left( \int_{B_{4r}(x_0)} |R_{\theta_k} - \bar{R}_{\theta_k}|^{p_1} dV_{\theta_k} \right)^{\frac{1}{p_1}} \left( \int_{B_{4r}(x_0)} dV_{\theta_k} \right)^{\frac{1}{n+1} - \frac{1}{p_1}} + \bar{R}_{\theta_k} \left( \int_{B_{4r}(x_0)} dV_{\theta_k} \right)^{\frac{1}{n+1}} \\ &\leq C_0^{\frac{1}{p_1}} \text{Vol}(B_r(x_0), \theta_k)^{\frac{1}{n+1} - \frac{1}{p_1}} + C_0 \text{Vol}(B_r(x_0), \theta_k)^{\frac{1}{n+1}} \rightarrow 0 \quad \text{as } r \rightarrow 0, \end{aligned}$$

where we have used Hölder's inequality, (4.8) and (4.11). By replacing  $r$  by a smaller radius, we can achieve  $\|P_k\|_{L^{n+1}(B_{4r}(x_0))} < \frac{2n+2}{nq} Y(S^{2n+1}, \theta_0)$ . Therefore, we can

apply Theorem 4.1(ii) and the assumption that  $\|u_k\|_{L^{2+\frac{2}{n}}(S^{2n+1})}^{2+\frac{2}{n}} = \text{Vol}(S^{2n+1}, \theta_k) =$

$\text{Vol}(S^{2n+1}, \theta_0)$  to conclude that  $u_k$  is bounded in  $L^q(B_{3r}(x_0))$  for any  $q > 2 + \frac{2}{n}$ , which implies that

$$(4.12) \quad \begin{aligned} & -(2 + \frac{2}{n}) \Delta_{\theta_0} u_k = R_{\theta_k} u_k^{1+\frac{2}{n}} - R_{\theta_0} u_k \\ &= (R_{\theta_k} - \bar{R}_{\theta_k}) u_k^{1+\frac{2}{n}} + \bar{R}_{\theta_k} u_k^{1+\frac{2}{n}} - R_{\theta_0} u_k \in L^p(B_{3r}(x_0)) \end{aligned}$$

for all  $2 + \frac{2}{n} < p < p_1$ . To see this, by Hölder's inequality we have

$$\begin{aligned} \|(R_{\theta_k} - \overline{R}_{\theta_k})u_k^{1+\frac{2}{n}}\|_{L^p(B_{3r}(x_0))}^p &= \int_{B_{3r}(x_0)} |R_{\theta_k} - \overline{R}_{\theta_k}|^p u_k^{(1+\frac{2}{n})p} dV_{\theta_0} \\ &\leq \left( \int_{B_{3r}(x_0)} |R_{\theta_k} - \overline{R}_{\theta_k}|^{p_1} u_k^{2+\frac{2}{n}} dV_{\theta_0} \right)^{\frac{p}{p_1}} \left( \int_{B_{3r}(x_0)} u_k^{q_0} dV_{\theta_0} \right)^{1-\frac{p}{p_1}} \\ &\leq \left( \int_{S^{2n+1}} |R_{\theta_k} - \overline{R}_{\theta_k}|^{p_1} dV_{\theta_k} \right)^{\frac{p}{p_1}} \|u_k\|_{L^{q_0}(B_{3r}(x_0))}^{q_0(\frac{p_1-p}{p_1})}. \end{aligned}$$

which is bounded because of (4.8) and the fact  $u_k$  is bounded in  $L^{q_0}(B_{3r}(x_0))$  where

$$\begin{aligned} q_0 &= \left[ \left(1 + \frac{2}{n}\right)p - \left(2 + \frac{2}{n}\right)\frac{p}{p_1} \right] \cdot \frac{p_1}{p_1 - p} \\ &\geq \left(1 + \frac{2}{n}\right)p - \left(2 + \frac{2}{n}\right)\frac{p}{p_1} > \left(2 + \frac{2}{n}\right) \left[ \left(1 + \frac{2}{n}\right) - \left(2 + \frac{2}{n}\right)\frac{1}{p_1} \right] \geq 2 + \frac{2}{n}, \end{aligned}$$

where we have used the assumption that  $p_1 > n + 1$ . On the other hand,

$$\begin{aligned} \|\overline{R}_{\theta_k} u_k^{1+\frac{2}{n}}\|_{L^p(B_{3r}(x_0))} &\leq \overline{R}_{\theta_k} \|u_k\|_{L^{p(1+\frac{2}{n})}(B_{3r}(x_0))}^{1+\frac{2}{n}} \text{ and} \\ \|R_{\theta_0} u_k\|_{L^p(B_{3r}(x_0))} &\leq |R_{\theta_0}| \|u_k\|_{L^p(B_{3r}(x_0))} \end{aligned}$$

are bounded by (4.8) and the fact  $u_k$  is bounded in  $L^q(B_{3r}(x_0))$  for all  $q > 2 + \frac{2}{n}$ . Combining all these, we prove (4.12). By (4.12) and the fact that  $u_k$  is bounded in  $L^{p_1}(B_{3r}(x_0))$ , we have

$$(4.13) \quad \|u_k\|_{S_2^p(B_{3r}(x_0))} \leq C \|\Delta_{\theta_0} u_k\|_{L^p(B_{3r}(x_0))} + C \|u_k\|_{L^p(B_{3r}(x_0))} \leq C(r),$$

where we have used Theorem 3.16 and 3.17 in [18]. See also [20].

Now assume that (4.10) is satisfied for every  $x \in S^{2n+1}$  and some radius  $r = r(x) > 0$ . Since  $S^{2n+1}$  is compact, the cover  $(B_{r(x)}(x))_{x \in S^{2n+1}}$  of  $S^{2n+1}$  admits a finite subcover  $B_{r_i}(x_i)$ , where  $r_i = r(x_i)$ ,  $1 \leq i \leq I$ . From (4.13), we then obtain the desired uniform bound

$$\|u_k\|_{S_2^p(S^{2n+1})} \leq I \max_{1 \leq i \leq I} C(r_i).$$

If (4.10) does not hold for every  $x$  with some  $r = r(x) > 0$ , we iteratively determine points  $x_l, l \in \mathbb{N}$ , and a subsequence  $\{u_k\}$  (reabeled) such that condition (4.9) is valid for any  $r > 0$ . This iteration terminates after finitely many steps. Indeed, given  $x_1, \dots, x_L$ , choose  $0 < r < \min_{i \neq j} d(x_i, x_j)/2$ . Then (4.9) yields the

bound

$$\begin{aligned}
L \cdot Y(S^{2n+1}, \theta_0)^{n+1} &\leq \sum_{i=1}^L \liminf_{k \rightarrow \infty} \int_{B_r(x_i)} |R_{\theta_k}|^{n+1} dV_{\theta_k} \\
&= \liminf_{k \rightarrow \infty} \int_{\cup_{i=1}^L B_r(x_i)} |R_{\theta_k}|^{n+1} dV_{\theta_k} \\
&\leq \sup_k \int_{S^{2n+1}} |R_{\theta_k}|^{n+1} dV_{\theta_k} \\
&\leq C \sup_k \left( \int_{S^{2n+1}} |R_{\theta_k} - \overline{R}_{\theta_k}|^{n+1} dV_{\theta_k} + \overline{R}_{\theta_k}^{n+1} \right) \\
&\leq C \sup_k \left( \int_{S^{2n+1}} |R_{\theta_k} - \overline{R}_{\theta_k}|^{p_1} dV_{\theta_k} + \overline{R}_{\theta_k}^{n+1} \right) < \infty
\end{aligned}$$

where we have used (4.8), Hölder's inequality, and  $\text{Vol}(S^{2n+1}, \theta_k) = \text{Vol}(S^{2n+1}, \theta_0)$ . By a covering argument as above we then obtain that  $\{u_k\}$  is bounded in  $S_2^p(S^{2n+1})$  on any compact subset of  $(S^{2n+1} \setminus \{x_1, \dots, x_L\}, \theta_0)$ .  $\square$

Now we can apply Theorem 4.2 to the solution of the flow equation (2.5).

**Lemma 4.3.** *For any time sequence  $t_k$ , we denote  $u_k = u(t_k)$  where  $u(t)$  is the solution to the flow equation (2.5). Then either*

- (i) *the sequence  $\{u_k\}$  is uniformly bounded in  $S_2^p(S^{2n+1}, \theta_0)$  for some  $p > n+1$ , or*
- (ii) *there exists a subsequence  $\{u_k\}$  (relabelled) and finitely many points  $x_1, \dots, x_L \in S^{2n+1}$  such that for any  $r > 0$  and any  $i \in \{1, \dots, L\}$  there holds*

$$(4.14) \quad \liminf_{k \rightarrow \infty} \left( \int_{B_r(x_i)} |R_{\theta(t_k)}|^{n+1} dV_{\theta(t_k)} \right)^{\frac{1}{n+1}} \geq Y(S^{2n+1}, \theta_0).$$

Moreover, the sequence  $\{u_k\}$  is bounded in  $S_2^p(S^{2n+1}, \theta_0)$  on any compact subset of  $(S^{2n+1} \setminus \{x_1, \dots, x_L\}, \theta_0)$ .

*Proof.* We are going to apply Theorem 4.2. Since the flow (2.5) keeps the volume fixed by Proposition 2.1, we only need to check (4.8) for some  $p_1 > n+1$ . First, note that

$$(4.15) \quad \overline{R}_{\theta(t_k)} = \frac{\int_{S^{2n+1}} R_{\theta(t_k)} dV_{\theta(t_k)}}{\int_{S^{2n+1}} dV_{\theta(t_k)}} = \frac{\alpha(t_k) \int_{S^{2n+1}} f dV_{\theta(t_k)}}{\int_{S^{2n+1}} dV_{\theta(t_k)}} \leq \alpha_2 \max_{S^{2n+1}} f$$

by (2.4) and Lemma 2.4. On the other hand, by Minkowski inequality, Proposition 2.1 and Lemma 2.4, we have

$$\begin{aligned}
&\left( \int_{S^{2n+1}} |R_{\theta(t_k)} - \overline{R}_{\theta(t_k)}|^{p_1} dV_{\theta(t_k)} \right)^{\frac{1}{p_1}} \\
&\leq F_{p_1}(t_k)^{\frac{1}{p_1}} + \alpha(t_k) \left( \int_{S^{2n+1}} f^{p_1} dV_{\theta(t_k)} \right)^{\frac{1}{p_1}} + \overline{R}_{\theta(t_k)} \text{Vol}(S^{2n+1}, \theta(t_k))^{\frac{1}{p_1}} \\
&= F_{p_1}(t_k)^{\frac{1}{p_1}} + (\alpha_2 \max_{S^{2n+1}} f + \overline{R}_{\theta(t_k)}) \text{Vol}(S^{2n+1}, \theta_0)^{\frac{1}{p_1}}
\end{aligned}$$

which is bounded by (4.15) and Lemma 3.1. Therefore, Lemma 4.3 follows from Theorem 4.2.  $\square$

*Remark.* One important thing we would like to mention is that concentration in the sense of (4.14) implies concentration of volume. To see this, by Hölder's inequality and Lemma 2.4, for any  $r > 0$  and  $p > n + 1$  we can estimate

$$\begin{aligned}
 (4.16) \quad & \left( \int_{B_r(x_i)} |R_{\theta(t_k)}|^{n+1} dV_{\theta(t_k)} \right)^{\frac{1}{n+1}} \\
 & \leq \alpha_2 \left( \max_{S^{2n+1}} f \right) \left( \int_{B_r(x_i)} dV_{\theta(t_k)} \right)^{\frac{1}{n+1}} + \left( \int_{B_r(x_i)} |\alpha(t_k)f - R_{\theta(t_k)}|^{n+1} dV_{\theta(t_k)} \right)^{\frac{1}{n+1}} \\
 & \leq \alpha_2 \left( \max_{S^{2n+1}} f \right) \left( \int_{B_r(x_i)} dV_{\theta(t_k)} \right)^{\frac{1}{n+1}} + \left( \int_{B_r(x_i)} |\alpha(t_k)f - R_{\theta(t_k)}|^p dV_{\theta(t_k)} \right)^{\frac{1}{p}} \left( \int_{B_r(x_i)} dV_{\theta(t_k)} \right)^{\frac{1}{n+1} - \frac{1}{p}}.
 \end{aligned}$$

It follows from (4.16) and Lemma 3.2 that if concentration of volume does not occur, then concentration in the sense of (4.14) does not occur.

Let  $\delta_n = \max_{S^{2n+1}} f / \min_{S^{2n+1}} f$ . By assumption (sbc) in Theorem 1.2, we have  $\delta_n < 2^{\frac{1}{n}}$ . Then there exists  $\epsilon_0 > 0$  such that

$$\frac{\delta_n^{\frac{n}{n+1}}}{2^{\frac{1}{n+1}}} = \frac{1 - \epsilon_0}{1 + \epsilon_0}.$$

In particular,  $(1 + \epsilon_0)\delta_n^{\frac{n}{n+1}} < 2^{\frac{1}{n+1}}$ . Set

$$(4.17) \quad \beta = (1 + \epsilon_0)Y(S^{2n+1}, \theta_0) \left( \min_{S^{2n+1}} f \right)^{-\frac{n}{n+1}}.$$

The next lemma is to estimate the number of blow-up points.

**Lemma 4.4.** *For any  $0 < u_0 \in C^\infty(S^{2n+1})$  with  $\int_{S^{2n+1}} u_0^{2+\frac{2}{n}} dV_{\theta_0} = \text{Vol}(S^{2n+1}, \theta_0)$  and  $E_f(u_0) \leq \beta$ , let  $u(t)$  be the solution of the flow (2.5) with the initial data  $u_0$ . If  $\{u(t_k)\}$  is a sequence with  $t_k \rightarrow \infty$  as  $k \rightarrow \infty$  and*

$$(4.18) \quad \max_{S^{2n+1}} f < 2^{\frac{1}{n}} \min_{S^{2n+1}} f,$$

then  $L = 1$ .

*Proof.* Suppose that  $x_1, \dots, x_L \in S^{2n+1}$  are the blow-up points. Let  $r = \min_{i \neq j} d(x_i, x_j)/2$ .

For any given  $\epsilon > 0$ , if  $k$  is sufficiently large, by (4.14) we have

$$\begin{aligned}
 (4.19) \quad & L(Y(S^{2n+1}, \theta_0) - \epsilon) \leq \sum_{i=1}^L \left( \int_{B_r(x_i)} |R_{\theta(t_k)}|^{n+1} dV_{\theta(t_k)} \right)^{\frac{1}{n+1}} \\
 & \leq L^{1-\frac{1}{n+1}} \left( \int_{S^{2n+1}} |R_{\theta(t_k)}|^{n+1} dV_{\theta(t_k)} \right)^{\frac{1}{n+1}} \\
 & \leq L^{1-\frac{1}{n+1}} \left( \int_{S^{2n+1}} |R_{\theta(t_k)} - \alpha(t_k)f|^{n+1} dV_{\theta(t_k)} \right)^{\frac{1}{n+1}} \\
 & \quad + L^{1-\frac{1}{n+1}} \alpha(t_k) \left( \int_{S^{2n+1}} f^{n+1} dV_{\theta(t_k)} \right)^{\frac{1}{n+1}}.
 \end{aligned}$$

Note that as  $k \rightarrow \infty$  the first term on the right hand side tends to 0 by Lemma 3.2. On the other hand, the second term can be estimated as follows:

$$\begin{aligned}
(4.20) \quad & \alpha(t_k) \left( \int_{S^{2n+1}} f^{n+1} dV_{\theta(t_k)} \right)^{\frac{1}{n+1}} \\
&= E_f(u(t_k)) \left( \int_{S^{2n+1}} f dV_{\theta(t_k)} \right)^{-\frac{1}{n+1}} \left( \int_{S^{2n+1}} f^{n+1} dV_{\theta(t_k)} \right)^{\frac{1}{n+1}} \\
&\leq E_f(u_0) \left( \max_{S^{2n+1}} f \right)^{\frac{n}{n+1}} \leq \beta \left( \max_{S^{2n+1}} f \right)^{\frac{n}{n+1}} \\
&= (1 + \epsilon_0) Y(S^{2n+1}, \theta_0) \left( \frac{\max_{S^{2n+1}} f}{\min_{S^{2n+1}} f} \right)^{\frac{n}{n+1}}
\end{aligned}$$

by (2.8), (2.9), (4.17), and Proposition 2.2. Combining (4.19) and (4.20), and using (4.18), we obtain

$$L(Y(S^{2n+1}, \theta_0) - \epsilon) \leq L^{1-\frac{1}{n+1}} (1 + \epsilon_0) Y(S^{2n+1}, \theta_0) \delta_n^{\frac{n}{n+1}}$$

for any  $\epsilon > 0$ . This implies that

$$L^{\frac{1}{n+1}} \leq (1 + \epsilon_0) \delta_n^{\frac{n}{n+1}} < 2^{\frac{1}{n+1}},$$

which implies  $L < 2$ . Since  $L$  is a natural number, one can easily conclude that  $L = 1$ .  $\square$

**Lemma 4.5.** *The blow-up point in Lemma 4.4 does not depend on the special choice of the sequence  $t_k$ .*

*Proof.* Suppose it were not true. Then there would exist  $x_1 \neq x_2$  in  $S^{2n+1}$  and two sequences  $\{t_j\}$  and  $\{t_k\}$  such that the sequences  $\{u(t_j)\}$  and  $\{u(t_k)\}$  are blow-up at  $x_1$  and  $x_2$  respectively. Then we define the new sequence  $\{u(t_l)\}$  such that  $u(t_{2j}) = u(t_j)$  and  $u(t_{2k+1}) = u(t_k)$ . That is, all the even terms of  $\{u(t_l)\}$  consist of the sequence  $\{u(t_j)\}$  while all the odd terms consist of the sequence  $\{u(t_k)\}$ . Then  $\{u(t_l)\}$  would be blow-up at  $x_1$  and  $x_2$ , which contradicts Lemma 4.4.  $\square$

**Lemma 4.6.** *Any sequence  $u_k = u(t_k)$  with  $t_k \rightarrow \infty$  is a Palais-Smale sequence of the energy functional  $E_f(u)$  if  $u$  is a solution of the flow (2.5) with a fixed initial data  $u_0$ .*

*Proof.* By Palais-Smale sequence we mean that  $\{u_k\}$  is bounded in  $S_1^2(S^{2n+1}, \theta_0)$  and  $dE_f(u_k) \rightarrow 0$  as  $k \rightarrow \infty$ . As  $k \rightarrow \infty$ ,

$$e_k = E_f(u_k) \rightarrow e_\infty$$

since  $E_f(u(t))$  is monotonic decreasing in time  $t$  by Proposition 2.2. Since  $E(u_k)$  is bounded by (2.14),  $\{u_k\}$  is bounded in  $S_1^2(S^{2n+1}, \theta_0)$ . On the other hand, for any



$\varphi \in S_1^2(S^{2n+1}, \theta_0) \hookrightarrow L^{2+\frac{2}{n}}(S^{2n+1}, \theta_0)$ , there holds

$$\begin{aligned} & \frac{1}{2} \left( \int_{S^{2n+1}} f u_k^{2+\frac{2}{n}} dV_{\theta_0} \right)^{\frac{n}{n+1}} |\langle DE_f(u_k), \varphi \rangle| \\ &= \left| \int_{S^{2n+1}} \left( (2 + \frac{2}{n}) \langle \nabla_{\theta_0} u_k, \nabla_{\theta_0} \varphi \rangle_{\theta_0} + R_{\theta_0} u_k \varphi \right) dV_{\theta_0} - \alpha(t_k) \int_{S^{2n+1}} f u_k^{1+\frac{2}{n}} \varphi dV_{\theta_0} \right| \\ &= \left| \int_{S^{2n+1}} (R_{\theta_k} - \alpha(t_k) f) u_k^{1+\frac{2}{n}} \varphi dV_{\theta_0} \right| \\ &\leq \left( \int_{S^{2n+1}} |R_{\theta_k} - \alpha(t_k) f|^{\frac{2n+2}{n+2}} u_k^{2+\frac{2}{n}} dV_{\theta_0} \right)^{\frac{n+2}{2n+2}} \left( \int_{S^{2n+1}} |\varphi|^{2+\frac{2}{n}} dV_{\theta_0} \right)^{\frac{n}{2n+2}} \rightarrow 0 \end{aligned}$$

as  $k \rightarrow \infty$  by Lemma 3.2.  $\square$

For every smooth positive function  $u(t)$ , set  $P(t) = \int_{S^{2n+1}} x u(t)^{2+\frac{2}{n}} dV_{\theta_0}$  where  $x = (x_1, \dots, x_{n+1}) \in S^{2n+1} \subset \mathbb{C}^{n+1}$ , and we define

$$(4.21) \quad \widehat{P}(t) = \frac{P(t)}{\|P(t)\|} \text{ if } \|P(t)\| \neq 0, \text{ otherwise } \widehat{P}(t) = P(t).$$

Clearly  $\widehat{P}(t) \in S^{2n+1}$  smoothly depends on the time  $t$  if  $u$  does. There exists a family of conformal CR diffeomorphisms  $\phi(t) : S^{2n+1} \rightarrow S^{2n+1}$  such that (see [21])

$$(4.22) \quad \int_{S^{2n+1}} x dV_h = (0, \dots, 0) \in \mathbb{C}^{n+1} \quad \text{for all } t > 0,$$

where the new contact form

$$(4.23) \quad h = h(t) = \phi(t)^*(\theta(t)) = v(t)^{2+\frac{2}{n}} \theta_0$$

is called the normalized contact form with  $v = v(t) = (u(t) \circ \phi(t)) |\det(d\phi(t))|^{\frac{n}{2n+2}}$  and the volume form  $dV_h = v(t)^{2+\frac{2}{n}} dV_{\theta_0}$ . In fact, the conformal CR diffeomorphism may be represented as  $\phi(t) = \phi_{p(t), r(t)} = \Psi \circ T_{p(t)} \circ D_{r(t)} \circ \pi$  for some  $p(t) \in \mathbb{H}^n$  and  $r(t) > 0$ . Here the CR diffeomorphism  $\pi : S^{2n+1} \setminus \{(0, \dots, 0, -1)\} \rightarrow \mathbb{H}^n$  is given by

$$(4.24) \quad \pi(x) = \left( \frac{x'}{1+x_{n+1}}, Re(\sqrt{-1} \frac{1-x_{n+1}}{1+x_{n+1}}) \right), x = (x', x_{n+1}) \in S^{2n+1},$$

where  $\mathbb{H}^n$  denotes the Heisenberg group, and  $D_\lambda, T_{(z', \tau')} : \mathbb{H}^n \rightarrow \mathbb{H}^n$  are respectively the dilation and translation on  $\mathbb{H}^n$  given by

$$(4.25) \quad D_\lambda(z, \tau) = (\lambda z, \lambda^2 \tau) \text{ and } T_{(z', \tau')}(z, \tau) = (z + z', \tau + \tau' + 2Im(z' \cdot \bar{z})) \text{ for } (z, \tau) \in \mathbb{H}^n.$$

And  $\Psi = \pi^{-1}$  is the inverse of  $\pi$ .

Meanwhile, the normalized function  $v$  satisfies

$$(4.26) \quad -\left(2 + \frac{2}{n}\right) \Delta_{\theta_0} v + R_{\theta_0} v = R_h v^{1+\frac{2}{n}},$$

where  $R_h = R_\theta \circ \phi(t)$  is the Webster scalar curvature of the normalized contact form  $h = h(t)$  in view of (4.23). Hereafter, we set  $f_\phi = f \circ \phi$ .

Now we state our main result of this section.

**Theorem 4.7.** *For any given  $u_0$  satisfying (2.2) with  $E_f(u_0) \leq \beta$ , consider the flow  $\theta(t)$  with initial data  $u_0$ . Let  $\{t_k\}$  be a time sequence of the flow with  $t_k \rightarrow \infty$  as  $k \rightarrow \infty$ . Let  $\{\theta_k\}$  be the corresponding contact forms such that  $\theta_k = u(t_k)^{\frac{2}{n}} \theta_0$ .*

Assume that  $\|R_{\theta_k} - R_\infty\|_{L^{p_1}(S^{2n+1}, \theta_k)} \rightarrow 0$  as  $k \rightarrow \infty$  for some  $p_1 > n + 1$  and a smooth function  $R_\infty > 0$  defined on  $S^{2n+1}$  which satisfies the simple bubble condition (sbc):

$$\frac{\max_{S^{2n+1}} R_\infty}{\min_{S^{2n+1}} R_\infty} < 2^{\frac{1}{n}}.$$

Then, up to a subsequence, either

(i)  $\{u_k\}$  is uniformly bounded in  $S_2^p(S^{2n+1}, \theta_0)$  for some  $p \in (n + 1, p_1)$ . Furthermore,  $u_k \rightarrow u_\infty$  in  $S_2^p(S^{2n+1}, \theta_0)$  as  $k \rightarrow \infty$ , where  $\theta_\infty = u_\infty^{\frac{2}{n}} \theta_0$  has Webster scalar curvature  $R_\infty$ , or

(ii) let  $h_k = \phi(t_k)^*(\theta_k) = v_k^{\frac{2}{n}} \theta_0$  be the associated sequence of the normalized contact forms satisfying  $\int_{S^{2n+1}} x dV_{h_k} = (0, \dots, 0) \in \mathbb{C}^{n+1}$ . Then, there exists  $Q \in S^{2n+1}$  such that

$$(4.27) \quad dV_{\theta_k} \rightharpoonup \text{Vol}(S^{2n+1}, \theta_0) \delta_Q, \quad \text{as } k \rightarrow \infty$$

in the weak sense of measures. In addition, for any  $\lambda \in (0, 1)$ , we have

$$(4.28) \quad v_k \rightarrow 1 \text{ in } C_P^{1, \lambda}(S^{2n+1}) \quad \text{as } k \rightarrow \infty.$$

Here  $C_P^{1, \lambda}(S^{2n+1})$  is the parabolic Hörmander Hölder spaces defined as in (2.26).

Due to the length of the proof of Theorem 4.7, we will divide the proof into several lemmas.

**Lemma 4.8.** *Suppose case (i) occurs. Then  $f$  can be realized as the Webster scalar curvature of some contact form conformal to  $\theta_0$ .*

*Proof.* If  $\{u_k\}$  is uniformly bounded in  $S_2^p(S^{2n+1}, \theta_0)$  for some  $p > n + 1$ , then up to a subsequence, there exists  $u_\infty \in S_2^p(S^{2n+1}, \theta_0)$  such that  $u_k \rightarrow u_\infty$  weakly in  $S_2^p(S^{2n+1}, \theta_0)$  and strongly in  $C^\sigma(S^{2n+1})$  for any  $0 < \sigma < (p - n - 1)/p$  as  $k \rightarrow \infty$ , where  $C^\sigma(S^{2n+1})$  is the standard Hölder space (see Theorem 3.16 and 3.17 in [18], and also [20]). Since  $\|R_{\theta_k} - R_\infty\|_{L^{p_1}(S^{2n+1}, \theta_k)} \rightarrow 0$  as  $k \rightarrow \infty$  and  $u_k$  satisfies

$$-(2 + \frac{2}{n}) \Delta_{\theta_0} u_k + R_{\theta_0} u_k = R_{\theta_k} u_k^{1 + \frac{2}{n}} \text{ on } S^{2n+1},$$

so  $u_\infty$  weakly solves

$$(4.29) \quad -(2 + \frac{2}{n}) \Delta_{\theta_0} u_\infty + R_{\theta_0} u_\infty = R_\infty u_\infty^{1 + \frac{2}{n}} \text{ on } S^{2n+1}.$$

By Theorem 3.22 in [18],  $u_\infty \in C^\infty(S^{2n+1})$  if  $R_\infty \in C^\infty(S^{2n+1})$ . Since  $\text{Vol}(S^{2n+1}, \theta_k) = \int_{S^{2n+1}} u_k^{2 + \frac{2}{n}} dV_{\theta_0} = \int_{S^{2n+1}} u_0^{2 + \frac{2}{n}} dV_{\theta_0} > 0$  by Proposition 2.1,  $\int_{S^{2n+1}} u_\infty^{2 + \frac{2}{n}} dV_{\theta_0} > 0$ . Since  $u_k > 0$ , we have  $u_\infty \geq 0$ . That is,  $u_\infty$  is nonnegative and not identically zero on  $S^{2n+1}$ . Applying Proposition A.1 in [28], we get  $u_\infty > 0$  on  $S^{2n+1}$ . Hence there exists constant  $C > 0$  such that

$$(4.30) \quad C^{-1} \leq u_\infty \leq C \text{ on } S^{2n+1}.$$

Moreover, we have

$$(4.31) \quad R_{\theta_k} \rightarrow R_\infty \text{ in } L^{p_1}(S^{2n+1}, \theta_0), \text{ as } k \rightarrow \infty,$$

and

$$(4.32) \quad u_k \rightarrow u_\infty \text{ in } S_2^p(S^{2n+1}, \theta_0), \text{ as } k \rightarrow \infty.$$

Hence, by (4.30), (4.32), and Lemma 3.2, we have

$$(4.33) \quad R_{\theta_k} - \alpha(t_k)f \rightarrow 0 \quad \text{in } L^p(S^{2n+1}, \theta_0).$$

On the other hand, by (4.30) and Hölder's inequality, we have

$$\begin{aligned} |\alpha(t_k) - \alpha(t_l)| \int_{S^{2n+1}} f^p dV_{\theta_0} &\leq \int_{S^{2n+1}} |R_{\theta_k} - \alpha(t_k)f|^p dV_{\theta_0} + \int_{S^{2n+1}} |R_{\theta_l} - \alpha(t_l)f|^p dV_{\theta_0} \\ &\quad + \left( \int_{S^{2n+1}} |R_{\theta_k} - R_\infty|^{p_1} dV_{\theta_0} \right)^{\frac{p_1}{p}} \left( \int_{S^{2n+1}} dV_{\theta_0} \right)^{\frac{p-p_1}{p}} \\ &\quad + \left( \int_{S^{2n+1}} |R_{\theta_l} - R_\infty|^{p_1} dV_{\theta_0} \right)^{\frac{p_1}{p}} \left( \int_{S^{2n+1}} dV_{\theta_0} \right)^{\frac{p-p_1}{p}} \end{aligned}$$

which tends to 0 as  $k, l \rightarrow \infty$  by (4.31) and (4.33). That is,  $\{\alpha(t_k)\}$  is a Cauchy sequence, which implies that  $\alpha(t_k) \rightarrow \alpha_\infty$  as  $k \rightarrow \infty$ . Combining all these, we have  $R_\infty = \alpha_\infty f$  for some  $\alpha_\infty > 0$ . Therefore, up to a constant multiple,  $u_\infty$  is a solution we want in view of (4.29).  $\square$

We are now ready to study case (ii), i.e. study the normalized flow  $v(t)$  defined in (4.23). For convenience, for each conformal CR diffeomorphism  $\phi$  from  $S^{2n+1}$  to itself, we denote  $(u \circ \phi)|\det(d\phi)|^{\frac{n}{2n+2}}$  by  $v$ . Note that  $v$  enjoys the following properties:

$$(4.34) \quad E(v) = E(u) \quad \text{and} \quad \int_{S^{2n+1}} v^{2+\frac{2}{n}} dV_{\theta_0} = \int_{S^{2n+1}} u^{2+\frac{2}{n}} dV_{\theta_0}.$$

**Lemma 4.9.** *There exists a constant  $C_0$  depending only on  $n$ , such that, for the normalized conformal factor  $v(t)$ , we have*

$$\int_{S^{2n+1}} v(t)^2 dV_{\theta_0} \geq C_0 > 0$$

for all  $t \geq 0$  with initial data in  $C_f^\infty$  which is defined in the proof.

*Proof.* It follows from (4.34) and Proposition 2.2 that

$$\begin{aligned} E(v(t)) &= E(u(t)) = E_f(u(t)) \left( \int_{S^{2n+1}} f u(t)^{2+\frac{2}{n}} dV_{\theta_0} \right)^{\frac{n}{n+1}} \\ &\leq E_f(u_0) \left( \int_{S^{2n+1}} f u(t)^{2+\frac{2}{n}} dV_{\theta_0} \right)^{\frac{n}{n+1}} \\ &\leq \beta \left( \left( \max_{S^{2n+1}} f \right) \int_{S^{2n+1}} u_0^{2+\frac{2}{n}} dV_{\theta_0} \right)^{\frac{n}{n+1}} \\ &= (1 + \epsilon_0) Y(S^{2n+1}, \theta_0) \text{Vol}(S^{2n+1}, \theta_0)^{\frac{n}{n+1}} \left( \frac{\max_{S^{2n+1}} f}{\min_{S^{2n+1}} f} \right)^{\frac{n}{n+1}} \\ &= (1 + \epsilon_0) R_{\theta_0} \text{Vol}(S^{2n+1}, \theta_0) \left( \frac{\max_{S^{2n+1}} f}{\min_{S^{2n+1}} f} \right)^{\frac{n}{n+1}} \\ &\leq (1 + \epsilon_0) R_{\theta_0} \text{Vol}(S^{2n+1}, \theta_0) \delta_n^{\frac{n}{n+1}}, \end{aligned}$$

where the initial data  $u_0$  satisfies

$$u_0 \in C_f^\infty := \{u \in C_*^\infty : u > 0 \text{ and } E_f(u) \leq \beta\}$$

with  $\beta$  defined as (4.17) and

$$C_*^\infty := \left\{ 0 < u \in C^\infty(S^{2n+1}) : \theta = u^{\frac{2}{n}} \theta_0 \text{ satisfies } \int_{S^{2n+1}} u^{2+\frac{2}{n}} dV_{\theta_0} = \int_{S^{2n+1}} dV_{\theta_0} \right\}.$$

Choose  $\epsilon = \frac{\epsilon_0}{2}(1 + \epsilon_0)^{-1} \delta_n^{-\frac{n}{n+1}} > 0$ , then

$$\begin{aligned} (1 + \epsilon_0)(2^{-\frac{1}{n+1}} + \epsilon) \delta_n^{\frac{n}{n+1}} &= (1 + \epsilon_0) 2^{-\frac{1}{n+1}} \delta_n^{\frac{n}{n+1}} + \epsilon(1 + \epsilon_0) \delta_n^{\frac{n}{n+1}} \\ &= (1 - \epsilon_0) + \frac{\epsilon_0}{2} = 1 - \frac{\epsilon_0}{2} < 1, \end{aligned}$$

thanks to  $\frac{\delta_n^{\frac{n}{n+1}}}{2^{\frac{1}{n+1}}} = \frac{1 - \epsilon_0}{1 + \epsilon_0}$ . By Lemma 6.1, there exists a constant  $C_\epsilon$  such that

$$Y(S^{2n+1}, \theta_0) \left( \int_{S^{2n+1}} v^{2+\frac{2}{n}} dV_{\theta_0} \right)^{\frac{n}{n+1}} \leq (2^{-\frac{1}{n+1}} + \epsilon) \left( 2 + \frac{2}{n} \right) \int_{S^{2n+1}} |\nabla_{\theta_0} v|_{\theta_0}^2 dV_{\theta_0} + C_\epsilon \int_{S^{2n+1}} v^2 dV_{\theta_0}.$$

By (4.34) and Proposition 2.1, we have

$$R_{\theta_0} \text{Vol}(S^{2n+1}, \theta_0) \leq (2^{-\frac{1}{n+1}} + \epsilon) \left( 2 + \frac{2}{n} \right) \int_{S^{2n+1}} |\nabla_{\theta_0} v|_{\theta_0}^2 dV_{\theta_0} + C_\epsilon \int_{S^{2n+1}} v^2 dV_{\theta_0}.$$

Combining the above inequalities, we obtain

$$(C_\epsilon - (2^{-\frac{1}{n+1}} + \epsilon) R_{\theta_0}) \int_{S^{2n+1}} v^2 dV_{\theta_0} \geq R_{\theta_0} \text{Vol}(S^{2n+1}, \theta_0) [1 - (1 + \epsilon_0)(2^{-\frac{1}{n+1}} + \epsilon) \delta_n^{\frac{n}{n+1}}]$$

which implies the desired estimate because of our choice of  $\epsilon$  above.  $\square$

The following definition appeared in [7], [9] and [16]: A sequence of positive functions  $v_k$  defined on  $S^{2n+1}$  satisfies condition (\*) if there is a set  $\Omega_k \subset S^{2n+1}$  with  $|\Omega_k| \geq C_1 > 0$ , and a constant  $C_2, \epsilon > 0$  such that

$$\int_{\Omega_k} v_k^{-\epsilon} dV_{\theta_0} \leq C_2,$$

where  $C_1, C_2$  are two constants independent of  $k$ . We have the following (see also Lemma 4.4 in [9]):

**Lemma 4.10.** *If condition (\*) does not hold with  $\epsilon > 0$  being small enough, then*

$$\int_{S^{2n+1}} v_k^\epsilon dV_{\theta_0} \rightarrow 0 \text{ as } k \rightarrow \infty.$$

*Proof.* We follow the proof of Theorem A.2 in [40]. By (4.26), we have

$$(4.35) \quad -(2 + \frac{2}{n}) \Delta_{\theta_0} v + R_{\theta_0} v = (R_h - \alpha f_\phi) v^{1+\frac{2}{n}} + \alpha f_\phi v^{1+\frac{2}{n}} \geq (R_h - \alpha f_\phi) v^{1+\frac{2}{n}}.$$

Let  $w = \log v - \beta$ , where  $\beta$  is a constant chosen such that  $\int_{S^{2n+1}} w dV_{\theta_0} = 0$ . Note that

$$|\nabla_{\theta_0} w|_{\theta_0}^2 + \Delta_{\theta_0} w = \frac{\Delta_{\theta_0} v}{v},$$

which together with (4.35) implies that

$$(4.36) \quad |\nabla_{\theta_0} w|_{\theta_0}^2 + \Delta_{\theta_0} w \leq \frac{R_{\theta_0}}{(2 + \frac{2}{n})} + \frac{(\alpha f_\phi - R_h) v^{\frac{2}{n}}}{(2 + \frac{2}{n})} = \frac{n^2}{4} + \frac{(\alpha f_\phi - R_h) v^{\frac{2}{n}}}{(2 + \frac{2}{n})}.$$

Integrating (4.36) over  $S^{2n+1}$ , we get

$$\begin{aligned}
 (4.37) \quad & \int_{S^{2n+1}} |\nabla_{\theta_0} w|_{\theta_0}^2 dV_{\theta_0} \\
 & \leq \frac{n^2}{4} \int_{S^{2n+1}} dV_{\theta_0} + \frac{n}{2n+2} \int_{S^{2n+1}} (\alpha f_\phi - R_h) v^{\frac{2}{n}} dV_{\theta_0} \\
 & \leq \frac{n^2}{4} \int_{S^{2n+1}} dV_{\theta_0} + \frac{n}{2n+2} \left( \int_{S^{2n+1}} |\alpha f_\phi - R_h|^{n+1} v^{2+\frac{2}{n}} dV_{\theta_0} \right)^{\frac{1}{n+1}} \left( \int_{S^{2n+1}} dV_{\theta_0} \right)^{\frac{n}{n+1}} \leq C_0
 \end{aligned}$$

by Hölder's inequality and Lemma 3.2. Now, for any  $p \in \mathbb{Z}^+$  multiplying (4.36) by  $|w|^{2p} \geq 0$  and integrating over  $S^{2n+1}$ , we get

$$\begin{aligned}
 (4.38) \quad & \int_{S^{2n+1}} (|w|^{2p} |\nabla_{\theta_0} w|_{\theta_0}^2 + |w|^{2p} \Delta_{\theta_0} w) dV_{\theta_0} \\
 & \leq \frac{n^2}{4} \int_{S^{2n+1}} |w|^{2p} dV_{\theta_0} + \frac{n}{2n+2} \int_{S^{2n+1}} (\alpha f_\phi - R_h) v^{\frac{2}{n}} |w|^{2p} dV_{\theta_0} \\
 & \leq \frac{n^2}{4} \int_{S^{2n+1}} |w|^{2p} dV_{\theta_0} + \frac{n}{2n+2} \left( \int_{S^{2n+1}} |\alpha f_\phi - R_h|^{n+1} v^{2+\frac{2}{n}} dV_{\theta_0} \right)^{\frac{1}{n+1}} \left( \int_{S^{2n+1}} |w|^{2p(\frac{n+1}{n})} dV_{\theta_0} \right)^{\frac{n}{n+1}} \\
 & \leq \frac{n^2}{4} \int_{S^{2n+1}} |w|^{2p} dV_{\theta_0} \\
 & \quad + \frac{n}{2n+2} F_{n+1}(t)^{\frac{1}{n+1}} Y(S^{2n+1}, \theta_0)^{-1} \left( \left(2 + \frac{2}{n}\right) \int_{S^{2n+1}} |\nabla_{\theta_0} w^p|_{\theta_0}^2 dV_{\theta_0} + R_{\theta_0} \int_{S^{2n+1}} |w|^{2p} dV_{\theta_0} \right)
 \end{aligned}$$

where we have used Hölder's inequality and Lemma 2.3. Since

$$\Delta_{\theta_0} |w|^{2p+1} = (2p+1) |w|^{2p} \Delta_{\theta_0} w + 2p(2p+1) |w|^{2p-1} |\nabla_{\theta_0} w|_{\theta_0}^2,$$

we can rewrite the left hand side of (4.38) as

$$\begin{aligned}
 (4.39) \quad & \int_{S^{2n+1}} (|w|^{2p} |\nabla_{\theta_0} w|_{\theta_0}^2 + |w|^{2p} \Delta_{\theta_0} w) dV_{\theta_0} \\
 & = \int_{S^{2n+1}} |w|^{2p} |\nabla_{\theta_0} w|_{\theta_0}^2 dV_{\theta_0} - 2p \int_{S^{2n+1}} |w|^{2p-1} |\nabla_{\theta_0} w|_{\theta_0}^2 dV_{\theta_0} \\
 & \geq \int_{S^{2n+1}} |w|^{2p} |\nabla_{\theta_0} w|_{\theta_0}^2 dV_{\theta_0} - \frac{1}{2} \int_{S^{2n+1}} |w|^{2p} |\nabla_{\theta_0} w|_{\theta_0}^2 dV_{\theta_0} - 2p^2 \int_{S^{2n+1}} |w|^{2p-2} |\nabla_{\theta_0} w|_{\theta_0}^2 dV_{\theta_0} \\
 & = \frac{1}{2(p+1)^2} \int_{S^{2n+1}} |\nabla_{\theta_0} w^{p+1}|_{\theta_0}^2 dV_{\theta_0} - 2 \int_{S^{2n+1}} |\nabla_{\theta_0} w^p|_{\theta_0}^2 dV_{\theta_0}
 \end{aligned}$$

where we have used Young's inequality. Combining (4.38) and (4.39), we obtain

$$\begin{aligned}
 (4.40) \quad & \frac{1}{2(p+1)^2} \int_{S^{2n+1}} |\nabla_{\theta_0} w^{p+1}|_{\theta_0}^2 dV_{\theta_0} \leq \left( F_{n+1}(t)^{\frac{1}{n+1}} Y(S^{2n+1}, \theta_0)^{-1} + 2 \right) \int_{S^{2n+1}} |\nabla_{\theta_0} w^p|_{\theta_0}^2 dV_{\theta_0} \\
 & \quad + \frac{n^2}{4} \left( F_{n+1}(t)^{\frac{1}{n+1}} Y(S^{2n+1}, \theta_0)^{-1} + 1 \right) \int_{S^{2n+1}} |w|^{2p} dV_{\theta_0}.
 \end{aligned}$$

By Lemma 3.2, there exists a constant  $C_1$  such that  $F_{n+1}(t)^{\frac{1}{n+1}} \leq C_1$  for all  $t \geq 0$ . Hence, it follows from (4.40) that

$$(4.41) \quad \|\nabla_{\theta_0} w^{p+1}\|_{L^2(S^{2n+1}, \theta_0)}^2 \leq (p+1)^2 (C_2 \|\nabla_{\theta_0} w^p\|_{L^2(S^{2n+1}, \theta_0)}^2 + C_3 \|w^p\|_{L^2(S^{2n+1}, \theta_0)}^2)$$

where  $C_2 = 2C_1 Y(S^{2n+1}, \theta_0)^{-1} + 4$  and  $C_3 = \frac{n^2}{2} (C_1 Y(S^{2n+1}, \theta_0)^{-1} + 1)$ . Recall the following Poincaré-type inequality (see Theorem 3.20 [18]): there exists a constant  $C_4$  such that

$$(4.42) \quad \|\varphi\|_{L^2(S^{2n+1}, \theta_0)} \leq C_4 \|\nabla_{\theta_0} \varphi\|_{L^2(S^{2n+1}, \theta_0)}$$

for all  $\varphi \in S_1^2(S^{2n+1}, \theta_0)$  satisfying  $\int_{S^{2n+1}} \varphi dV_{\theta_0} = 0$ . It follows that with  $A := \max\{\sqrt{C_0}, \sqrt{C_3}\}$  and  $B := \sqrt{\frac{C_2}{C_3}} + 2C_4$  for all  $p \in \mathbb{Z}^+$  there holds

$$(4.43) \quad \|\nabla_{\theta_0} w^{p+1}\|_{L^2(S^{2n+1}, \theta_0)} \leq A(p+1) (B \|\nabla_{\theta_0} w^p\|_{L^2(S^{2n+1}, \theta_0)} + \|w^p\|_{L^2(S^{2n+1}, \theta_0)}).$$

To see this, if we denote  $\overline{w^p} = \frac{1}{\text{Vol}(S^{2n+1}, \theta_0)} \int_{S^{2n+1}} w^p dV_{\theta_0}$ , by (4.41) and (4.42) we have

$$\begin{aligned} & \|\nabla_{\theta_0} w^{p+1}\|_{L^2(S^{2n+1}, \theta_0)} \\ & \leq (p+1) \left( \sqrt{C_2} \|\nabla_{\theta_0} w^p\|_{L^2(S^{2n+1}, \theta_0)} + \sqrt{C_3} \|w^p\|_{L^2(S^{2n+1}, \theta_0)} \right) \\ & \leq (p+1) \left( \sqrt{C_2} \|\nabla_{\theta_0} w^p\|_{L^2(S^{2n+1}, \theta_0)} + \sqrt{C_3} \|w^p - \overline{w^p}\|_{L^2(S^{2n+1}, \theta_0)} + \sqrt{C_3} \|\overline{w^p}\|_{L^2(S^{2n+1}, \theta_0)} \right) \\ & \leq (p+1) \left( (\sqrt{C_2} + \sqrt{C_3} C_4) \|\nabla_{\theta_0} w^p\|_{L^2(S^{2n+1}, \theta_0)} + \sqrt{\frac{C_3}{\text{Vol}(S^{2n+1}, \theta_0)}} \left| \int_{S^{2n+1}} w^p dV_{\theta_0} \right| \right) \\ & \leq (p+1) \left( (\sqrt{C_2} + \sqrt{C_3} C_4) \|\nabla_{\theta_0} w^p\|_{L^2(S^{2n+1}, \theta_0)} + \sqrt{C_3} \|w^p\|_{L^2(S^{2n+1}, \theta_0)} \right) \\ & = \sqrt{C_3} (p+1) \left( \left( \sqrt{\frac{C_2}{C_3}} + C_4 \right) \|\nabla_{\theta_0} w^p\|_{L^2(S^{2n+1}, \theta_0)} + \|w^p\|_{L^2(S^{2n+1}, \theta_0)} \right). \end{aligned}$$

By induction on  $p$  we can now obtain the estimates

$$(4.44) \quad \|\nabla_{\theta_0} w^p\|_{L^2(S^{2n+1}, \theta_0)} \leq A^p B^{p-1} p^p \quad \text{and} \quad \|w^p\|_{L^2(S^{2n+1}, \theta_0)} \leq A^p B^p p^p$$

for all  $p \in \mathbb{Z}^+$ . Indeed, for  $p = 1$ , (4.44) follows from (4.37), (4.42) and the fact that  $A \geq \sqrt{C_0}$  and  $B \geq C_4$ . Moreover, assuming that (4.44) is true for some  $p$ . Then from (4.43) and Hölder's inequality we deduce

$$\begin{aligned} \|\nabla_{\theta_0} w^{p+1}\|_{L^2(S^{2n+1}, \theta_0)} & \leq A(p+1) (B \|\nabla_{\theta_0} w^p\|_{L^2(S^{2n+1}, \theta_0)} + \|w^p\|_{L^2(S^{2n+1}, \theta_0)}) \\ & \leq A(p+1) (B \cdot A^p B^{p-1} p^p + A^p B^p p^p) = A^{p+1} B^p 2(p+1) p^p. \end{aligned}$$

Now the first inequality in (4.44) for  $p+1$  follows from this inequality, because Bernoulli's inequality says that  $(1+t)^{\frac{1}{p}} \leq 1 + \frac{t}{p}$  for all  $t > 0$ , which implies that  $2 \leq ((p+1)/p)^p$ . Similarly, by (4.42), we have

$$\|w^{p+1} - \overline{w^{p+1}}\|_{L^2(S^{2n+1}, \theta_0)} \leq C_4 \|\nabla_{\theta_0} w^{p+1}\|_{L^2(S^{2n+1}, \theta_0)},$$

which implies that

$$\begin{aligned}
\|w^{p+1}\|_{L^2(S^{2n+1}, \theta_0)} &\leq C_4 \|\nabla_{\theta_0} w^{p+1}\|_{L^2(S^{2n+1}, \theta_0)} + \|\overline{w^{p+1}}\|_{L^2(S^{2n+1}, \theta_0)} \\
&= C_4 \|\nabla_{\theta_0} w^{p+1}\|_{L^2(S^{2n+1}, \theta_0)} + \frac{1}{\sqrt{\text{Vol}(S^{2n+1}, \theta_0)}} \left| \int_{S^{2n+1}} w^{p+1} dV_{\theta_0} \right| \\
&\leq C_4 \|\nabla_{\theta_0} w^{p+1}\|_{L^2(S^{2n+1}, \theta_0)} + \|w^{p+1}\|_{L^1(S^{2n+1}, \theta_0)} \\
&\leq C_4 \|\nabla_{\theta_0} w^{p+1}\|_{L^2(S^{2n+1}, \theta_0)} + \|w^p\|_{L^2(S^{2n+1}, \theta_0)} \|w\|_{L^2(S^{2n+1}, \theta_0)} \\
&\leq C_4 A^{p+1} B^p (p+1)^{p+1} + A^p B^p p^p \cdot AB \\
&\leq \frac{1}{2} A^{p+1} B^{p+1} (p+1)^{p+1} + A^{p+1} B^{p+1} p^p \\
&\leq A^{p+1} B^{p+1} (p+1)^{p+1}
\end{aligned}$$

since  $2C_4 \leq B$ . This completes the induction step.

Now using Stirling's formula to estimate  $p! \geq \left(\frac{p}{e}\right)^p$  for any  $p \in \mathbb{Z}^+$ , for any  $p_0 > 0$  we can estimate

$$\int_{S^{2n+1}} (e^{p_0|w|} - 1) dV_{\theta_0} \leq \sum_{p=1}^{\infty} \int_{S^{2n+1}} \left( \frac{p_0 e|w|}{p} \right)^p dV_{\theta_0} \leq \sum_{p=1}^{\infty} (p_0 e AB)^p,$$

which is finite whenever  $p_0 e AB < 1$ . Choosing  $p_0 = \frac{1}{2eAB} > 0$ , we then conclude that

$$(4.45) \quad \int_{S^{2n+1}} v^{-p_0} dV_{\theta_0} \cdot \int_{S^{2n+1}} v^{p_0} dV_{\theta_0} \leq \left( \int_{S^{2n+1}} \exp(p_0|w|) dV_{\theta_0} \right)^2 \leq C.$$

The inequality (4.45) says that  $v_k$  does not satisfy condition (\*) with  $\epsilon = p_0$  if and only if  $\int_{S^{2n+1}} v_k^{p_0} dV_{\theta_0} \rightarrow 0$  as  $k \rightarrow \infty$ .  $\square$

**Corollary 4.11.** *The normalized conformal factor  $v(t)$  satisfies condition (\*).*

*Proof.* For  $0 < \delta < 2$ , let  $\epsilon = \frac{(2+2n)\delta}{2+n\delta} > 0$ . Then by (4.34), Hölder's inequality, Proposition 2.1, and Lemma 4.9, we have

$$\begin{aligned}
0 < C_0 &\leq \int_{S^{2n+1}} v(t)^2 dV_{\theta_0} \leq \left( \int_{S^{2n+1}} v(t)^{\frac{2+2n}{n}} dV_{\theta_0} \right)^{\frac{n(2-\delta)}{2+2n}} \left( \int_{S^{2n+1}} v(t)^{\delta \cdot \frac{2+2n}{2+n\delta}} dV_{\theta_0} \right)^{\frac{2+n\delta}{2+2n}} \\
&= \text{Vol}(S^{2n+1}, \theta_0)^{\frac{n(2-\delta)}{2+2n}} \left( \int_{S^{2n+1}} v(t)^{\epsilon} dV_{\theta_0} \right)^{\frac{2+n\delta}{2+2n}}.
\end{aligned}$$

Hence, it follows from Lemma 4.10 that  $v(t)$  satisfies condition (\*) with the  $\epsilon > 0$  defined above.  $\square$

**Lemma 4.12.** *There exists a uniform constant  $C_3 > 0$  such that  $v(t) \geq C_3 > 0$  on  $S^{2n+1}$  for all  $t \geq 0$  and any  $u_0 \in C_f^\infty$ .*

*Proof.* Note that  $v(t)$  satisfies (4.26). For  $\delta > 0$ , multiply (4.26) by  $v^{-1-2\delta}$ , then integrate both sides, we get

$$\begin{aligned}
(4.46) \quad & \left(2 + \frac{2}{n}\right) \int_{S^{2n+1}} |\nabla_{\theta_0} v^{-\delta}|_{\theta_0}^2 dV_{\theta_0} \\
&= -\frac{\delta^2}{1+2\delta} \int_{S^{2n+1}} R_h v^{1+\frac{2}{n}-1-2\delta} dV_{\theta_0} + \frac{\delta^2 R_{\theta_0}}{1+2\delta} \int_{S^{2n+1}} v^{-2\delta} dV_{\theta_0} \\
&= -\frac{\delta^2}{1+2\delta} \int_{S^{2n+1}} (R_h - \alpha f_\phi) v^{\frac{2-2n\delta}{n}} dV_{\theta_0} - \frac{\delta^2}{1+2\delta} \int_{S^{2n+1}} \alpha f_\phi v^{\frac{2-2n\delta}{n}} dV_{\theta_0} \\
&\quad + \frac{\delta^2 R_{\theta_0}}{1+2\delta} \int_{S^{2n+1}} v^{-2\delta} dV_{\theta_0} \\
&\leq C(\delta, n) \|R_h - f_\phi\|_{L^{\frac{2n+2}{2-2n\delta}}(S^{2n+1}, h)} + \frac{\delta^2 R_{\theta_0}}{1+2\delta} \int_{S^{2n+1}} v^{-2\delta} dV_{\theta_0}
\end{aligned}$$

where we have used the short form  $f_\phi$  for  $f \circ \phi(t)$ .

Let  $\lambda_1 = n/2$  be the first nonzero eigenvalue of  $-\Delta_{\theta_0}$ , then for any  $\varphi \in C^\infty(S^{2n+1})$ , Raleigh's inequality gives

$$\int_{S^{2n+1}} \varphi^2 dV_{\theta_0} \leq \frac{1}{\text{Vol}(S^{2n+1}, \theta_0)} \left( \int_{S^{2n+1}} \varphi dV_{\theta_0} \right)^2 + \lambda_1^{-1} \int_{S^{2n+1}} |\nabla_{\theta_0} \varphi|_{\theta_0}^2 dV_{\theta_0}.$$

Choose  $\varphi = v^{-\delta}$ , then

$$(4.47) \quad \int_{S^{2n+1}} v^{-2\delta} dV_{\theta_0} \leq \frac{1}{\text{Vol}(S^{2n+1}, \theta_0)} \left( \int_{S^{2n+1}} v^{-\delta} dV_{\theta_0} \right)^2 + \lambda_1^{-1} \int_{S^{2n+1}} |\nabla_{\theta_0} v^{-\delta}|_{\theta_0}^2 dV_{\theta_0}.$$

It follows from Corollary 4.11 that  $v(t)$  satisfies condition (\*) with some  $\epsilon > 0$ . By condition (\*), if  $\delta < \epsilon$ , we have

$$\int_{S^{2n+1}} v^{-\delta} dV_{\theta_0} = \int_{\Omega} v^{-\delta} dV_{\theta_0} + \int_{\Omega^c} v^{-\delta} dV_{\theta_0} \leq C_2 + \left( \int_{\Omega^c} v^{-2\delta} dV_{\theta_0} \right)^{\frac{1}{2}} \left( \int_{\Omega^c} dV_{\theta_0} \right)^{\frac{1}{2}}.$$

This and Young's inequality imply that, for any  $\eta > 0$ , there holds

$$\begin{aligned}
(4.48) \quad & \frac{1}{\text{Vol}(S^{2n+1}, \theta_0)} \left( \int_{S^{2n+1}} v^{-\delta} dV_{\theta_0} \right)^2 \leq \frac{1}{\text{Vol}(S^{2n+1}, \theta_0)} \left[ C_2 + \left( \int_{\Omega^c} v^{-2\delta} dV_{\theta_0} \right)^{\frac{1}{2}} \left( \int_{\Omega^c} dV_{\theta_0} \right)^{\frac{1}{2}} \right]^2 \\
& \leq \frac{(1+\eta^{-1})C_2^2}{\text{Vol}(S^{2n+1}, \theta_0)} + \frac{(1+\eta)|\Omega^c|}{\text{Vol}(S^{2n+1}, \theta_0)} \left( \int_{\Omega^c} v^{-2\delta} dV_{\theta_0} \right).
\end{aligned}$$

Since  $\text{Vol}(\Omega, \theta_0) > 0$ , then

$$\frac{\text{Vol}(\Omega^c, \theta_0)}{\text{Vol}(S^{2n+1}, \theta_0)} = 1 - 2\theta \text{ where } \theta = \frac{\text{Vol}(\Omega, \theta_0)}{2\text{Vol}(S^{2n+1}, \theta_0)} > 0.$$

Thus choosing  $\eta$  sufficiently small such that  $(1+\eta)(1-2\theta) < 1-\theta$ , together with (4.46), (4.47) and (4.48), we conclude that

$$\begin{aligned}
\int_{S^{2n+1}} v^{-2\delta} dV_{\theta_0} &\leq C + (1-\theta) \int_{S^{2n+1}} v^{-2\delta} dV_{\theta_0} + \frac{n^2 \delta^2}{4\lambda_1(1+2\delta)} \int_{S^{2n+1}} v^{-2\delta} dV_{\theta_0} \\
&\quad + C \|R_h - f_\phi\|_{L^{\frac{2n+2}{2-2n\delta}}(S^{2n+1}, h)}.
\end{aligned}$$



By Lemma 3.2 and taking  $\delta > 0$  sufficiently small, we have

$$(4.49) \quad \int_{S^{2n+1}} v^{-2\delta} dV_{\theta_0} \leq C.$$

Now let  $G(Q, \cdot)$  be the Green's function for  $-\Delta_{\theta_0}$  with singularity at  $Q \in S^{2n+1}$ . Then from [37], we have

$$(4.50) \quad G(Q, \cdot) > 0 \text{ and } \|G(Q, \cdot)\|_{L^p(S^{2n+1}, \theta_0)} \leq C(p) \text{ for all } p < 2.$$

Hence, for  $\gamma > 0$  sufficiently small so that  $2 + \gamma < 2 + \frac{2}{n}$ , we obtain by (4.26) and (4.50) that

$$\begin{aligned} v^{-\gamma}(Q) &= \frac{1}{\text{Vol}(S^{2n+1}, \theta_0)} \int_{S^{2n+1}} v^{-\gamma} dV_{\theta_0} - \int_{S^{2n+1}} G(Q, \cdot) \Delta_{\theta_0}(v^{-\gamma}) dV_{\theta_0} \\ &= \frac{1}{\text{Vol}(S^{2n+1}, \theta_0)} \int_{S^{2n+1}} v^{-\gamma} dV_{\theta_0} - \frac{n}{2n+2} \int_{S^{2n+1}} G(Q, \cdot) \\ &\quad \left( \gamma R_h v^{\frac{2}{n}-\gamma} - \gamma R_{\theta_0} v^{-\gamma} + \left(2 + \frac{2}{n}\right) \gamma (1 + \gamma) v^{-\gamma-2} |\nabla_{\theta_0} v|_{\theta_0}^2 \right) dV_{\theta_0} \\ &\leq \frac{1}{\text{Vol}(S^{2n+1}, \theta_0)} \int_{S^{2n+1}} v^{-\gamma} dV_{\theta_0} + \frac{n}{2n+2} \gamma R_{\theta_0} \int_{S^{2n+1}} G(Q, \cdot) v^{-\gamma} dV_{\theta_0} \\ &\quad - \frac{n}{2n+2} \gamma \int_{S^{2n+1}} G(Q, \cdot) (R_h - \alpha f_{\phi}) v^{\frac{2}{n}-\gamma} dV_{\theta_0} \\ &\leq \frac{1}{\text{Vol}(S^{2n+1}, \theta_0)} \int_{S^{2n+1}} v^{-\gamma} dV_{\theta_0} \\ &\quad + \frac{n}{2n+2} \gamma R_{\theta_0} C \left( \frac{n+1}{n} \right) \|v^{-\gamma}\|_{L^{n+1}(S^{2n+1}, \theta_0)} \\ &\quad + \frac{n}{2n+2} \gamma C \left( \frac{2n+2}{2n-n\gamma} \right) \|R_h - \alpha f_{\phi}\|_{L^{\frac{2n+2}{2-n\gamma}}(S^{2n+1}, h)}. \end{aligned}$$

Finally, choosing  $\gamma = \frac{2\delta}{n+1}$  with  $\delta < 1$  and using (4.49), (4.50) and Lemma 3.2, we get the desired lower bound  $v \geq C_3 > 0$ .  $\square$

**Lemma 4.13.** *Suppose that case (i) does not occur in Theorem 4.7. For any sequence  $\{t_k\}$ , let  $\lambda_1(\theta(t_k))$  be the first non-zero eigenvalue of the sub-Laplacian with respect to the contact form  $\theta(t_k) = u^{\frac{2}{n}}(t_k)\theta_0$ . Then there exists a subsequence  $\{t_j\}$  such that  $\lambda_1(\theta(t_j)) \geq \beta_0 > 0$  for all  $j$ , where  $\beta_0$  is a positive constant independent of  $j$ .*

*Proof.* Since  $\theta(t_k) = u^{\frac{2}{n}}(t_k)\theta_0$  is diffeomorphic to the normalized contact form  $h(t_k) = v_k^{\frac{2}{n}}\theta_0$ , we only need to get the first non-zero eigenvalue estimate for  $h(t_k) = v_k^{\frac{2}{n}}\theta_0$ . By Lemma 4.3, for any sequence  $\{t_k\}$ , we have

$$(4.51) \quad \liminf_{k \rightarrow \infty} \left( \frac{1}{\text{Vol}(S^{2n+1}, \theta_0)} \int_{B_r(Q)} |R_{h(t_k)}|^{n+1} dV_{h(t_k)} \right)^{\frac{1}{n+1}} \geq \frac{n(n+1)}{2}$$

for any  $r > 0$ . Therefore for each  $k$ , there exists  $r_k > 0$  such that

$$(4.52) \quad \left( \frac{1}{\text{Vol}(S^{2n+1}, \theta_0)} \int_{B_{r_k}(Q)} |R_{h(t_k)}|^{n+1} dV_{h(t_k)} \right)^{\frac{1}{n+1}} = \frac{n(n+1)}{2},$$

since  $|R_{h(t_k)}|^{n+1}$  is integrable with respect to  $dV_{h(t_k)}$ . It follows from (4.51) and (4.52) that  $r_k \rightarrow 0$  as  $k \rightarrow \infty$ .

Now, up to a rotation, we may assume  $Q = S = (0, \dots, 0, -1)$ , the south pole of  $S^{2n+1}$ . Let  $\Psi(z, \tau) = \pi^{-1}(z, \tau)$  for  $(z, \tau) \in \mathbb{H}^n \subset \mathbb{C}^n \times \mathbb{R}$  where  $\pi$  is defined as in (4.24). Set

$$w_k(z, \tau) = |\det(d\Psi)|^{\frac{n}{2n+2}} (v_k \circ \Psi) \geq 0$$

to obtain a sequence  $w_k : \mathbb{H}^n \rightarrow \mathbb{R}$  which is in  $S_2^p(\mathbb{H}^n)$  locally and converges to a function such that

$$(2 + \frac{2}{n})\Delta_{\mathbb{H}^n} w_\infty + \alpha(\infty)f(Q)w_\infty^{\frac{n}{2n+2}} = 0.$$

Here  $\Delta_{\mathbb{H}^n}$  is the sub-Laplacian with respect to the standard contact form  $\theta_{\mathbb{H}^n} = d\tau + i \sum_{j=1}^n (z_j d\bar{z}_j - \bar{z}_j dz_j)$  on  $\mathbb{H}^n$  (see [18]). By the classification theorem of Jerison and Lee in [29], up to the scaling with  $\alpha(\infty)f(Q) = n(n+1)/2$ ,

$$w_\infty(z, \tau) = \lambda^n \omega \circ D_\lambda \circ T_{(z', \tau')}(z, \tau) \text{ and } \omega(z, \tau) = \left( \frac{4}{\tau^2 + (1 + |z|^2)^2} \right)^{\frac{n}{2}},$$

where  $D_\lambda, T_{(z', \tau')} : \mathbb{H}^n \rightarrow \mathbb{H}^n$  are respectively the dilation and translation on  $\mathbb{H}^n$  defined as in (4.25). Then let us define

$$\psi = \lambda^n \Psi \circ D_\lambda \circ T_{(z', \tau')} \circ \pi : S^{2n+1} \setminus \{S\} \rightarrow S^{2n+1} \setminus \{S\}.$$

It follows directly from the definition that

$$\tilde{h}(t_k) := \psi^*(h(t_k)) \rightarrow \theta_0 \text{ as } k \rightarrow \infty$$

locally in  $S_2^p(S^{2n+1} \setminus \{S\}, \theta_0)$  for any  $p < \infty$ . Denote by  $\tilde{w}_k^{\frac{2}{n}} \theta_0$  the contact form  $\tilde{h}(t_k) = \psi^*(h(t_k))$ . Hence we have

$$(4.53) \quad \tilde{w}_k \rightarrow 1 \text{ as } k \rightarrow \infty \text{ locally in } S_2^p(S^{2n+1} \setminus \{S\}, \theta_0).$$

Now for any given  $0 < \rho < 1$ , let  $\zeta_\rho$  be a cut-off function on  $\mathbb{H}^n$  defined by

$$\zeta_\rho(z, \tau) = \log \log \frac{1}{\rho} - \log \log |(z, \tau)|_{\mathbb{H}^n}$$

for  $\rho^{-1/e} < |(z, \tau)|_{\mathbb{H}^n} = \sqrt[4]{|z|^4 + \tau^2} < \rho^{-1}$ ; and with  $\zeta_\rho(z) = 0$  in  $\mathbb{H}^n \setminus B_{\rho^{-1}}(0)$ ,  $\zeta_\rho(z) = 1$  in  $B_{\rho^{-1/e}}(0)$ . Then we have

$$(4.54) \quad \begin{aligned} & \int_{S^{2n+1}} |\nabla_{\theta_0}(\zeta_\rho \circ \pi)|_{\theta_0}^{2n+2} dV_{\theta_0} = \int_{\mathbb{H}^n} |\nabla_{\theta_{\mathbb{H}^n}} \zeta_\rho|_{\theta_{\mathbb{H}^n}}^{2n+2} dV_{\theta_{\mathbb{H}^n}} \\ & = C \iint_{\{\rho^{-1/e} < \sqrt[4]{|z|^4 + \tau^2} < \rho^{-1}\}} \left( \frac{|z|^3}{(|z|^4 + \tau^2) \log(|z|^4 + \tau^2)} \right)^{2n+2} dz d\tau \\ & \rightarrow 0 \text{ as } \rho \rightarrow 0. \end{aligned}$$

For any  $k$ , let  $\psi_k$  be the first eigenfunction of the sub-Laplacian of  $\tilde{h}(t_k)$  with nonzero first eigenvalue  $\lambda_1(\tilde{h}(t_k))$ . We normalize the eigenfunction such that

$$(4.55) \quad \int_{S^{2n+1}} \psi_k dV_{\tilde{h}(t_k)} = 0 \text{ and } \int_{S^{2n+1}} \psi_k^2 dV_{\tilde{h}(t_k)} = 1.$$

We claim that there exists a subsequence  $k$  such that  $\lambda_1(\tilde{h}(t_k)) \geq \frac{n}{4}$ . Assume the claim does not hold, we would have  $\lambda_1(\tilde{h}(t_k)) \leq \frac{n}{4}$  for all sufficiently large  $k$ . This

would imply

$$(4.56) \quad \int_{S^{2n+1}} |\nabla_{\tilde{h}(t_k)} \psi_k|_{\tilde{h}(t_k)}^2 dV_{\tilde{h}(t_k)} = \lambda_1(\tilde{h}(t_k)) \leq \frac{n}{4}$$

by (4.53). Note that, by Lemma 4.12,  $\tilde{w}_k \geq c_0 > 0$  for some constant  $c_0 > 0$  depending only on  $C_3$  and  $\lambda$ . Therefore for each fixed  $\rho > 0$  sufficiently small, by (4.55) we have

$$\begin{aligned} \int_{S^{2n+1}} (\psi_k \zeta_\rho \circ \pi)^2 dV_{\theta_0} &\leq c_0^{-(2+\frac{2}{n})} \int_{S^{2n+1}} (\psi_k \zeta_\rho \circ \pi)^2 \tilde{w}_k^{2+\frac{2}{n}} dV_{\theta_0} \\ &\leq c_0^{-(2+\frac{2}{n})} \int_{S^{2n+1}} \psi_k^2 dV_{\tilde{h}(t_k)} = c_0^{-(2+\frac{2}{n})} \end{aligned}$$

since  $(\zeta_\rho \circ \pi)^2 \leq 1$  for all  $z$ . We also have

$$\begin{aligned} \int_{S^{2n+1}} |\nabla_{\theta_0} (\psi_k \zeta_\rho \circ \pi)|^2 dV_{\theta_0} &\leq c_0^{-2} \int_{S^{2n+1}} |\nabla_{\theta_0} (\psi_k \zeta_\rho \circ \pi)|_{\theta_0}^2 \tilde{w}_k^2 dV_{\theta_0} \\ &= 2c_0^{-2} \left( \int_{S^{2n+1}} |\nabla_{\theta_0} \psi_k|_{\theta_0}^2 \tilde{w}_k^2 (\zeta_\rho \circ \pi)^2 dV_{\theta_0} + \int_{S^{2n+1}} |\nabla_{\theta_0} (\zeta_\rho \circ \pi)|_{\theta_0}^2 \tilde{w}_k^2 \psi_k^2 dV_{\theta_0} \right). \end{aligned}$$

Observe that the first term in bracket of the right hand side is bounded since  $(\zeta_\rho \circ \pi)^2 \leq 1$  and

$$(4.57) \quad \int_{S^{2n+1}} |\nabla_{\theta_0} \psi_k|_{\theta_0}^2 \tilde{w}_k^2 dV_{\theta_0} = \int_{S^{2n+1}} |\nabla_{\tilde{h}(t_k)} \psi_k|_{\tilde{h}(t_k)}^2 dV_{\tilde{h}(t_k)} \leq \frac{n}{4}.$$

For the second term, we can apply Hölder's inequality to get

$$\begin{aligned} \int_{S^{2n+1}} |\nabla_{\theta_0} (\zeta_\rho \circ \pi)|_{\theta_0}^2 \tilde{w}_k^2 \psi_k^2 dV_{\theta_0} \\ \leq \left( \int_{S^{2n+1}} |\nabla_{\theta_0} (\zeta_\rho \circ \pi)|_{\theta_0}^{2n+2} dV_{\theta_0} \right)^{\frac{1}{n+1}} \left( \int_{S^{2n+1}} |\tilde{w}_k \psi_k|^{2+\frac{2}{n}} dV_{\theta_0} \right)^{\frac{n}{n+1}}. \end{aligned}$$

The first factor is bounded thanks to (4.54), while the second factor can be estimated as follows:

$$\begin{aligned} (4.58) \quad &Y(S^{2n+1}, \theta_0) \left( \int_{S^{2n+1}} |\tilde{w}_k \psi_k|^{2+\frac{2}{n}} dV_{\theta_0} \right)^{\frac{n}{n+1}} \\ &= Y(S^{2n+1}, \theta_0) \|\tilde{w}_k \psi_k\|_{L^{2+\frac{2}{n}}(S^{2n+1}, \theta_0)}^2 = Y(S^{2n+1}, \theta_0) \left( \int_{S^{2n+1}} |\psi_k|^{2+\frac{2}{n}} dV_{\tilde{h}(t_k)} \right)^{\frac{n}{n+1}} \\ &\leq (2 + \frac{2}{n}) \int_{S^{2n+1}} |\nabla_{\tilde{h}(t_k)} \psi_k|_{\tilde{h}(t_k)}^2 dV_{\tilde{h}(t_k)} + \int_{S^{2n+1}} R_{\tilde{h}(t_k)} \psi_k^2 dV_{\tilde{h}(t_k)} \\ &= (2 + \frac{2}{n}) \int_{S^{2n+1}} |\nabla_{\tilde{h}(t_k)} \psi_k|_{\tilde{h}(t_k)}^2 dV_{\tilde{h}(t_k)} + \alpha(t_k) \int_{S^{2n+1}} f \psi_k^2 dV_{\tilde{h}(t_k)} \\ &\quad + \int_{S^{2n+1}} (R_{\tilde{h}(t_k)} - \alpha(t_k) f) \psi_k^2 dV_{\tilde{h}(t_k)} \\ &\leq (2 + \frac{2}{n}) \int_{S^{2n+1}} |\nabla_{\tilde{h}(t_k)} \psi_k|_{\tilde{h}(t_k)}^2 dV_{\tilde{h}(t_k)} + \alpha(t_k) \int_{S^{2n+1}} f \psi_k^2 dV_{\tilde{h}(t_k)} \\ &\quad + \left( \int_{S^{2n+1}} |R_{\tilde{h}(t_k)} - \alpha(t_k) f|^{n+1} dV_{\tilde{h}(t_k)} \right)^{\frac{1}{n+1}} \left( \int_{S^{2n+1}} |\psi_k|^{2+\frac{2}{n}} dV_{\tilde{h}(t_k)} \right)^{\frac{n}{n+1}} \end{aligned}$$

where the first inequality follows from Lemma 2.3 and the last inequality follows from Hölder's inequality. When  $k$  is sufficiently large,

$$\left( \int_{S^{2n+1}} |R_{\tilde{h}(t_k)} - \alpha(t_k)f|^{n+1} dV_{\tilde{h}(t_k)} \right)^{\frac{1}{n+1}} \leq \frac{Y(S^{2n+1}, \theta_0)}{2}$$

by Lemma 3.2. Combining this with (4.58), we deduce that

$$\begin{aligned} & \frac{Y(S^{2n+1}, \theta_0)}{2} \left( \int_{S^{2n+1}} |\psi_k|^{2+\frac{2}{n}} dV_{\tilde{h}(t_k)} \right)^{\frac{n}{n+1}} \\ & \leq (2 + \frac{2}{n}) \int_{S^{2n+1}} |\nabla_{\tilde{h}(t_k)} \psi_k|_{\tilde{h}(t_k)}^2 dV_{\tilde{h}(t_k)} + \alpha(t_k) \int_{S^{2n+1}} f \psi_k^2 dV_{\tilde{h}(t_k)} \\ & \leq (2 + \frac{2}{n}) \int_{S^{2n+1}} |\nabla_{\tilde{h}(t_k)} \psi_k|_{\tilde{h}(t_k)}^2 dV_{\tilde{h}(t_k)} + \alpha_2 \left( \max_{S^{2n+1}} f \right) \int_{S^{2n+1}} \psi_k^2 dV_{\tilde{h}(t_k)} \leq C \end{aligned}$$

by (4.55), (4.57), and Lemma 2.4.

Therefore we have shown that  $\{\psi_k \zeta_\rho \circ \pi\}$  is a bound sequence in  $S_1^2(S^{2n+1}, \theta_0)$ . Thus there exists a subsequence, still denote by  $\{\psi_k \zeta_\rho \circ \pi\}$ , and a function  $\psi_\rho$  such that  $\psi_k \zeta_\rho \circ \pi \rightarrow \psi_\rho$  as  $k \rightarrow \infty$  weakly in  $S_1^2(S^{2n+1}, \theta_0)$  which is also strongly convergent in  $L^q(S^{2n+1}, \theta_0)$  with  $q < \frac{n}{2n+2}$ . Then  $\{\psi_\rho\}$  is bounded in  $S_1^2(S^{2n+1}, \theta_0)$  since  $\|\psi_\rho\|_{S_1^2(S^{2n+1}, \theta_0)} \leq \lim_{k \rightarrow \infty} \|\psi_k \zeta_\rho \circ \pi\|_{S_1^2(S^{2n+1}, \theta_0)}$ . Therefore there exists a subsequence  $\{\psi_{\rho_j}\}$  and  $\psi_0 \in S_1^2(S^{2n+1}, \theta_0)$  such that  $\psi_{\rho_j} \rightarrow \psi_0$  as  $j \rightarrow \infty$  weakly in  $S_1^2(S^{2n+1}, \theta_0)$ .

Now we claim that if we let  $\pi = \Psi^{-1}$ , then there hold

$$(4.59) \quad \lim_{\rho \rightarrow 0} \lim_{k \rightarrow \infty} \|\psi_k \zeta_\rho \circ \pi - \psi_k\|_{L^2(S^{2n+1}, \tilde{h}(t_k))} = 0,$$

and

$$(4.60) \quad \lim_{j \rightarrow \infty} \lim_{k \rightarrow \infty} (\|\nabla_{\tilde{h}(t_k)}(\psi_k \zeta_{\rho_j} \circ \pi)\|_{L^2(S^{2n+1}, \tilde{h}(t_k))} - \|\nabla_{\tilde{h}(t_k)} \psi_k\|_{L^2(S^{2n+1}, \tilde{h}(t_k))}) \leq 0.$$

If we assume these, then we have

$$\begin{aligned} \int_{S^{2n+1}} \psi_0^2 dV_{\theta_0} &= \lim_{j \rightarrow \infty} \int_{S^{2n+1}} \psi_{\rho_j}^2 dV_{\theta_0} = \lim_{j \rightarrow \infty} \lim_{k \rightarrow \infty} \int_{S^{2n+1}} (\psi_k \zeta_{\rho_j} \circ \pi)^2 dV_{\theta_0} \\ &= \lim_{k \rightarrow \infty} \int_{S^{2n+1}} \psi_k^2 \tilde{w}_k^{2+\frac{2}{n}} dV_{\theta_0} = 1 \end{aligned}$$

where we have used (4.53), (4.55) and (4.59). Similarly, we have

$$\begin{aligned} \int_{S^{2n+1}} \psi_0 dV_{\theta_0} &= \lim_{j \rightarrow \infty} \int_{S^{2n+1}} \psi_{\rho_j} dV_{\theta_0} = \lim_{j \rightarrow \infty} \lim_{k \rightarrow \infty} \int_{S^{2n+1}} \psi_k \zeta_{\rho_j} \circ \pi dV_{\theta_0} \\ &= \lim_{k \rightarrow \infty} \int_{S^{2n+1}} \psi_k \tilde{w}_k^{2+\frac{2}{n}} dV_{\theta_0} = 0. \end{aligned}$$

Thus  $\psi_0 \not\equiv 0$  and Raleigh's inequality gives

$$(4.61) \quad \|\nabla_{\theta_0} \psi_0\|_{L^2(S^{2n+1}, \theta_0)} \geq \frac{n}{2}.$$

Hence, we get the following contradiction:

$$\begin{aligned}
\frac{n}{4} &\geq \lim_{k \rightarrow \infty} \lambda_1(\tilde{h}(t_k)) = \lim_{k \rightarrow \infty} \int_{S^{2n+1}} |\nabla_{\tilde{h}(t_k)} \psi_k|_{\tilde{h}(t_k)}^2 dV_{\tilde{h}(t_k)} \\
&\geq \lim_{j \rightarrow \infty} \lim_{k \rightarrow \infty} \|\nabla_{\tilde{h}(t_k)}(\psi_k \zeta_{\rho_j} \circ \pi)\|_{L^2(S^{2n+1}, \tilde{h}(t_k))} = \lim_{j \rightarrow \infty} \lim_{k \rightarrow \infty} \|\nabla_{\theta_0}(\psi_k \zeta_{\rho_j} \circ \pi)\|_{L^2(S^{2n+1}, \theta_0)} \\
&\geq \lim_{j \rightarrow \infty} \|\nabla_{\theta_0} \psi_{\rho_j}\|_{L^2(S^{2n+1}, \theta_0)} \geq \|\nabla_{\theta_0} \psi_0\|_{L^2(S^{2n+1}, \theta_0)} \geq \frac{n}{2},
\end{aligned}$$

where the first inequality follows from (4.56), the second inequality follows from (4.60), the second equality follows from (4.53), and the last inequality follows from (4.61).

Therefore it remains to show (4.59) and (4.60). By Hölder's inequality and the fact that  $\zeta_\rho \circ \pi \leq 1$ , we have

$$\begin{aligned}
\int_{S^{2n+1}} |\psi_k(\zeta_\rho \circ \pi - 1)|^2 dV_{\tilde{h}(t_k)} &\leq \int_{\{x \in S^{2n+1} | \zeta_\rho \circ \pi < 1\}} |\psi_k|^2 dV_{\tilde{h}(t_k)} \\
&\leq \text{Vol}(\{x \in S^{2n+1} | \zeta_\rho \circ \pi < 1\}, \tilde{h}(t_k))^{\frac{1}{n+1}} \left( \int_{S^{2n+1}} |\psi_k|^{2+\frac{2}{n}} dV_{\tilde{h}(t_k)} \right)^{\frac{n}{n+1}},
\end{aligned}$$

where the first factor can be estimated by (4.53):<sup>1</sup>

$$\begin{aligned}
\text{Vol}(\{x \in S^{2n+1} | \zeta_\rho \circ \pi < 1\}, \tilde{h}(t_k)) &= \text{Vol}(\{x \in S^{2n+1} | \zeta_\rho \circ \pi < 1\}, \tilde{w}(t_k)^{\frac{2}{n}} \theta_0) \\
&\xrightarrow{k \rightarrow \infty} \text{Vol}(\{x \in S^{2n+1} | \zeta_\rho \circ \pi < 1\}, \theta_0) \\
&= \int_{\mathbb{H}^n \setminus B_{\rho^{-1/e}}(0)} \left( \frac{4}{(1+|z|^2)^2 + \tau^2} \right)^{n+1} dV_{\theta_{\mathbb{H}^n}} \xrightarrow{\rho \rightarrow 0} 0
\end{aligned}$$

and the second factor can be bounded as follows:

$$\begin{aligned}
\left( \int_{S^{2n+1}} |\psi_k|^{2+\frac{2}{n}} dV_{\tilde{h}(t_k)} \right)^{\frac{n}{n+1}} &\leq C \left( \int_{S^{2n+1}} |\psi_k|^{2+\frac{2}{n}} dV_{\theta_0} \right)^{\frac{n}{n+1}} \\
&\leq C \|\psi_k\|_{S_1^2(S^{2n+1}, \theta_0)} \leq C \|\psi_k\|_{S_1^2(S^{2n+1}, \tilde{h}(t_k))} \leq C
\end{aligned}$$

by (4.53), (4.55), (4.56) and Lemma 2.3. This proves (4.59). To prove (4.60), we apply Hölder's inequality to get

$$\begin{aligned}
&\|\psi_k \nabla_{\tilde{h}(t_k)}(\zeta_\rho \circ \pi)\|_{L^2(S^{2n+1}, \tilde{h}(t_k))}^2 \\
(4.63) \quad &= \int_{S^{2n+1}} |\nabla_{\tilde{h}(t_k)}(\zeta_\rho \circ \pi)|_{\tilde{h}(t_k)}^2 \psi_k^2 dV_{\tilde{h}(t_k)} \\
&\leq \left( \int_{S^{2n+1}} |\nabla_{\tilde{h}(t_k)}(\zeta_\rho \circ \pi)|_{\tilde{h}(t_k)}^{2n+2} dV_{\tilde{h}(t_k)} \right)^{\frac{1}{n+1}} \left( \int_{S^{2n+1}} \psi_k^{2+\frac{2}{n}} dV_{\tilde{h}(t_k)} \right)^{\frac{n}{n+1}},
\end{aligned}$$

<sup>1</sup>The integral can be estimated as follows:

$$\begin{aligned}
&\int_{\mathbb{H}^n \setminus B_{\rho^{-1}}(0)} \frac{dz d\tau}{(\tau^2 + (1+|z|^2)^2)^{n+1}} = \int_{\{\sqrt{\tau^2 + |z|^4} \geq \rho^{-1}\}} \frac{dz d\tau}{(\tau^2 + (1+|z|^2)^2)^{n+1}} \\
&\leq 2 \int_{\{|z| \geq \rho^{-1}\}} \left( \int_{\sqrt{\rho^{-4} - |z|^4}}^{\infty} \frac{d\tau}{1 + \tau^2} \right) \frac{dz}{(1+|z|^2)^{2n}} = 2 \int_{\{|z| \geq \rho^{-1}\}} \left[ \tan^{-1}(\tau) \right]_0^{\infty} \frac{dz}{(1+|z|^2)^{2n}} \\
&\leq \pi \int_{\{|z| \geq \rho^{-1}\}} \frac{dz}{(1+|z|^2)^{2n}} = C \int_{\rho^{-1}}^{\infty} \frac{r^{2n-1} dr}{(1+r^2)^{2n}} \leq C \int_{\rho^{-1}}^{\infty} \frac{dr}{r^{2n+1}} = O(\rho^{2n}).
\end{aligned}$$

where the first factor can be estimated as follows:

$$(4.64) \quad \int_{S^{2n+1}} |\nabla_{\tilde{h}(t_k)}(\zeta_\rho \circ \pi)|_{\tilde{h}(t_k)}^{2n+2} dV_{\tilde{h}(t_k)} = \int_{S^{2n+1}} |\nabla_{\theta_0}(\zeta_\rho \circ \pi)|_{\theta_0}^{2n+2} dV_{\theta_0} \xrightarrow{\rho \rightarrow 0} 0$$

by (4.54), and the second factor is bounded by (4.62). Hence, as  $\rho \rightarrow 0$ , we have

$$\begin{aligned} & \|\nabla_{\tilde{h}(t_k)}(\psi_k \zeta_\rho \circ \pi)\|_{L^2(S^{2n+1}, \tilde{h}(t_k))} \\ & \leq \|(\zeta_\rho \circ \pi) \nabla_{\tilde{h}(t_k)} \psi_k\|_{L^2(S^{2n+1}, \tilde{h}(t_k))} + \|\psi_k \nabla_{\tilde{h}(t_k)}(\zeta_\rho \circ \pi)\|_{L^2(S^{2n+1}, \tilde{h}(t_k))} \\ & \leq \|\nabla_{\tilde{h}(t_k)} \psi_k\|_{L^2(S^{2n+1}, \tilde{h}(t_k))} + o(1), \end{aligned}$$

by (4.58), (4.64) and the fact that  $\zeta_\rho \circ \pi \leq 1$ . This proves (4.60).  $\square$

Next we have the following estimate, which is used to control the upper bound of the normalized conformal factor  $v$ .

**Lemma 4.14.** *On  $(S^{2n+1}, \theta_0)$ , if  $h = v^{\frac{2}{n}} \theta_0$  with  $v > 0$  is a contact form satisfying*

- (i)  $\int_{S^{2n+1}} v^{2+\frac{2}{n}} dV_{\theta_0} = \int_{S^{2n+1}} u_0^{2+\frac{2}{n}} dV_{\theta_0} = \text{Vol}(S^{2n+1}, \theta_0)$ ,
- (ii)  $\int_{S^{2n+1}} |R_h|^p v^{2+\frac{2}{n}} dV_{\theta_0} \leq a_p \text{Vol}(S^{2n+1}, \theta_0)$  for some  $p > n+1$ ,
- (iii)  $0 < \Lambda \leq \lambda_1(h)$ , where  $\lambda_1(h)$  is the first nonzero eigenvalue of the sub-Laplacian  $-\Delta_h$ , where  $a_p, \Lambda$  are positive constants and
- (iv) assuming in addition the condition: There exists some positive constants  $\sigma_0, l_0$  such that

$$(4.65) \quad \int_{\{x \in S^{2n+1} | v(x) \geq \sigma_0\}} dV_{\theta_0} \geq l_0.$$

Then there exists  $\epsilon_0 > 0$  and a constant  $C_0$  depending only on  $n, p, a_p, \Lambda, \sigma_0, l_0$  with

$$\int_{S^{2n+1}} v^{2+\frac{2}{n}+\epsilon_0} dV_{\theta_0} \leq C_0.$$

*Proof.* Since

$$-(2 + \frac{2}{n}) \Delta_{\theta_0} v + R_{\theta_0} v = R_h v^{1+\frac{2}{n}} \text{ on } S^{2n+1},$$

we multiply it by  $v^\beta$  for some  $\beta > 0$  and integrate it to get

$$(2 + \frac{2}{n})\beta \int_{S^{2n+1}} v^{\beta-1} |\nabla_{\theta_0} v|_{\theta_0}^2 dV_{\theta_0} + R_{\theta_0} \int_{S^{2n+1}} v^{\beta+1} dV_{\theta_0} = \int_{S^{2n+1}} R_h v^{1+\frac{2}{n}+\beta} dV_{\theta_0}.$$

Set  $w = v^{\frac{1+\beta}{2}}$ , then

$$(4.66) \quad (2 + \frac{2}{n}) \int_{S^{2n+1}} |\nabla_{\theta_0} w|_{\theta_0}^2 dV_{\theta_0} = -\frac{(\beta+1)^2}{4\beta} R_{\theta_0} \int_{S^{2n+1}} w^2 dV_{\theta_0} + \frac{(\beta+1)^2}{4\beta} \int_{S^{2n+1}} R_h v^{\frac{2}{n}} w^2 dV_{\theta_0}.$$

By Lemma 2.3, we have

$$\begin{aligned} & \frac{n(n+1)}{2} \text{Vol}(S^{2n+1}, \theta_0)^{\frac{1}{n+1}} \left( \int_{S^{2n+1}} w^{2+\frac{2}{n}} dV_{\theta_0} \right)^{\frac{n}{n+1}} \\ (4.67) \quad & \leq \int_{S^{2n+1}} \left( (2 + \frac{2}{n}) |\nabla_{\theta_0} w|_{\theta_0}^2 + R_{\theta_0} w^2 \right) dV_{\theta_0} \\ & = \left( 1 - \frac{(\beta+1)^2}{4\beta} \right) R_{\theta_0} \int_{S^{2n+1}} w^2 dV_{\theta_0} + \frac{(\beta+1)^2}{4\beta} \int_{S^{2n+1}} R_h v^{\frac{2}{n}} w^2 dV_{\theta_0} \end{aligned}$$

where we have used (4.66). For a sufficiently large number  $b > 0$ , we get

$$(4.68) \quad b^p \int_{\{|R_h| > b\}} v^{2+\frac{2}{n}} dV_{\theta_0} \leq \int_{S^{2n+1}} |R_h|^p v^{2+\frac{2}{n}} dV_{\theta_0} \leq a_p \text{Vol}(S^{2n+1}, \theta_0)$$

by assumption (ii). By Hölder's inequality, we have

$$(4.69) \quad \begin{aligned} & \int_{\{|R_h| > b\}} |R_h|^p v^{\frac{2}{n}} w^2 dV_{\theta_0} \\ & \leq \left( \int_{S^{2n+1}} |R_h|^p v^{2+\frac{2}{n}} dV_{\theta_0} \right)^{\frac{1}{p}} \left( \int_{\{|R_h| > b\}} v^{2+\frac{2}{n}} dV_{\theta_0} \right)^{\frac{1}{n+1}-\frac{1}{p}} \left( \int_{S^{2n+1}} w^{2+\frac{2}{n}} dV_{\theta_0} \right)^{\frac{n}{n+1}} \\ & \leq a_p^{\frac{1}{p}} \left( \frac{a_p}{b^p} \right)^{\frac{1}{n+1}-\frac{1}{p}} \text{Vol}(S^{2n+1}, \theta_0)^{\frac{1}{n+1}} \left( \int_{S^{2n+1}} w^{2+\frac{2}{n}} dV_{\theta_0} \right)^{\frac{n}{n+1}} \end{aligned}$$

where we have used (4.68) and the assumption (ii).

On the other hand, for any  $\psi \in S_1^2(S^{2n+1}, h)$ , Raleigh's inequality gives

$$\int_{S^{2n+1}} \psi^2 dV_h \leq \frac{1}{\text{Vol}(S^{2n+1}, \theta_0)} \left( \int_{S^{2n+1}} \psi dV_h \right)^2 + \lambda_1(h)^{-1} \int_{S^{2n+1}} |\nabla_h \psi|_h^2 dV_h.$$

Now let  $\psi = v^\epsilon$  with  $0 < \epsilon < 1$ , then

$$(4.70) \quad \int_{S^{2n+1}} v^{2+\frac{2}{n}+2\epsilon} dV_{\theta_0} \leq \frac{1}{\text{Vol}(S^{2n+1}, \theta_0)} \left( \int_{S^{2n+1}} v^{2+\frac{2}{n}+\epsilon} dV_{\theta_0} \right)^2 + \Lambda^{-1} \int_{S^{2n+1}} v^2 |\nabla_{\theta_0} v^\epsilon|_{\theta_0}^2 dV_{\theta_0}$$

by assumption (iii). Notice that condition (4.65) implies that there exist  $\sigma_0 > 0$  and  $l_0 > 0$  such that

$$(4.71) \quad \int_{E_{\sigma_0}} dV_{\theta_0} \geq l_0 \text{ i.e. } \text{Vol}(E_{\sigma_0}, \theta_0) \geq l_0,$$

where  $E_{\sigma_0} = \{x \in S^{2n+1} | v(x) \geq \sigma_0\}$ . Observe that

$$\begin{aligned} & \int_{S^{2n+1}} v^{2+\frac{2}{n}+\epsilon} dV_{\theta_0} = \int_{E_{\sigma_0}} v^{2+\frac{2}{n}+\epsilon} dV_{\theta_0} + \int_{E_{\sigma_0}^c} v^{2+\frac{2}{n}+\epsilon} dV_{\theta_0} \\ & = \int_{E_{\sigma_0}} (v^{2+\frac{2}{n}} - \sigma_0^{2+\frac{2}{n}}) v^\epsilon dV_{\theta_0} + \sigma_0^{2+\frac{2}{n}} \int_{E_{\sigma_0}} v^\epsilon dV_{\theta_0} + \int_{E_{\sigma_0}^c} v^{2+\frac{2}{n}+\epsilon} dV_{\theta_0} \\ & \leq \left( \int_{E_{\sigma_0}} (v^{2+\frac{2}{n}} - \sigma_0^{2+\frac{2}{n}}) v^{2\epsilon} dV_{\theta_0} \right)^{\frac{1}{2}} \left( \int_{E_{\sigma_0}} (v^{2+\frac{2}{n}} - \sigma_0^{2+\frac{2}{n}}) dV_{\theta_0} \right)^{\frac{1}{2}} \\ & \quad + \sigma_0^{2+\frac{2}{n}} \left( \int_{E_{\sigma_0}} v^{2+\frac{2}{n}} dV_{\theta_0} \right)^{\frac{n\epsilon}{2n+2}} \text{Vol}(S^{2n+1}, \theta_0)^{1-\frac{n\epsilon}{2n+2}} + \sigma_0^{2+\frac{2}{n}+\epsilon} \int_{E_{\sigma_0}^c} dV_{\theta_0} \\ & \leq \left( \int_{E_{\sigma_0}} v^{2+\frac{2}{n}+2\epsilon} dV_{\theta_0} \right)^{\frac{1}{2}} \left( \int_{E_{\sigma_0}} (v^{2+\frac{2}{n}} - \sigma_0^{2+\frac{2}{n}}) dV_{\theta_0} \right)^{\frac{1}{2}} + C(\sigma_0) \end{aligned}$$

where we have used assumption (i). This together with Young's inequality implies that for any  $0 < \eta < 1$

$$\begin{aligned}
(4.72) \quad & \left( \int_{S^{2n+1}} v^{2+\frac{2}{n}+\epsilon} dV_{\theta_0} \right)^2 \\
& \leq (1+\eta) \left( \int_{E_{\sigma_0}} v^{2+\frac{2}{n}+2\epsilon} dV_{\theta_0} \right) \left( \int_{E_{\sigma_0}} (v^{2+\frac{2}{n}} - \sigma_0^{2+\frac{2}{n}}) dV_{\theta_0} \right) + (1+\eta^{-1})C(\sigma_0)^2 \\
& \leq (1+\eta) \left( \int_{E_{\sigma_0}} v^{2+\frac{2}{n}+2\epsilon} dV_{\theta_0} \right) \left( \text{Vol}(S^{2n+1}, \theta_0) - \sigma_0^{2+\frac{2}{n}} \text{Vol}(E_{\sigma_0}, \theta_0) \right) + (1+\eta^{-1})C(\sigma_0)^2
\end{aligned}$$

by assumption (i). By choosing  $\sigma_0$  small enough, we can assume  $\sigma_0^{2+\frac{2}{n}} \text{Vol}(E_{\sigma_0}, \theta_0) < \text{Vol}(S^{2n+1}, \theta_0)$ . Since  $\sigma_0^{2+\frac{2}{n}} \text{Vol}(E_{\sigma_0}, \theta_0) \geq \sigma_0^{2+\frac{2}{n}} l_0 > 0$ , then we may choose sufficiently small  $\eta > 0$  such that

$$(4.73) \quad (1+\eta) \left( \text{Vol}(S^{2n+1}, \theta_0) - \sigma_0^{2+\frac{2}{n}} l_0 \right) \leq (1-\delta) \text{Vol}(S^{2n+1}, \theta_0)$$

for some positive constant  $\delta = \delta(\sigma_0, l_0)$ . Then

$$\begin{aligned}
\int_{S^{2n+1}} v^{2+\frac{2}{n}+2\epsilon} dV_{\theta_0} & \leq \frac{1}{\text{Vol}(S^{2n+1}, \theta_0)} \left( \int_{S^{2n+1}} v^{2+\frac{2}{n}+\epsilon} dV_{\theta_0} \right)^2 + \Lambda^{-1} \int_{S^{2n+1}} v^2 |\nabla_{\theta_0} v^\epsilon|_{\theta_0}^2 dV_{\theta_0} \\
& \leq \frac{(1+\eta) \left( \text{Vol}(S^{2n+1}, \theta_0) - \sigma_0^{2+\frac{2}{n}} \text{Vol}(E_{\sigma_0}, \theta_0) \right)}{\text{Vol}(S^{2n+1}, \theta_0)} \int_{S^{2n+1}} v^{2+\frac{2}{n}+2\epsilon} dV_{\theta_0} \\
& \quad + \frac{(1+\eta^{-1})C(\sigma_0)^2}{\text{Vol}(S^{2n+1}, \theta_0)} + \Lambda^{-1} \int_{S^{2n+1}} v^2 |\nabla_{\theta_0} v^\epsilon|_{\theta_0}^2 dV_{\theta_0} \\
& \leq (1-\delta) \int_{S^{2n+1}} v^{2+\frac{2}{n}+2\epsilon} dV_{\theta_0} + \frac{(1+\eta^{-1})C(\sigma_0)^2}{\text{Vol}(S^{2n+1}, \theta_0)} + \Lambda^{-1} \int_{S^{2n+1}} v^2 |\nabla_{\theta_0} v^\epsilon|_{\theta_0}^2 dV_{\theta_0}
\end{aligned}$$

where we have used (4.70) in the first inequality, (4.72) in the second inequality, and (4.73) in the last inequality. This implies that

$$(4.74) \quad \delta \int_{S^{2n+1}} v^{2+\frac{2}{n}+2\epsilon} dV_{\theta_0} \leq C(\sigma_0, l_0) + \Lambda^{-1} \int_{S^{2n+1}} v^2 |\nabla_{\theta_0} v^\epsilon|_{\theta_0}^2 dV_{\theta_0}.$$

Choose  $\beta = 1 + 2\epsilon$ , then  $w = v^{\frac{1+\beta}{2}} = v^{1+\epsilon}$ . Then by (4.74)

$$\begin{aligned}
(4.75) \quad & \int_{S^{2n+1}} v^{2+\frac{2}{n}+2\epsilon} dV_{\theta_0} \leq C\delta^{-1} + (\delta\Lambda)^{-1} \int_{S^{2n+1}} v^2 |\nabla_{\theta_0} v^\epsilon|_{\theta_0}^2 dV_{\theta_0} \\
& = C\delta^{-1} + (\delta\Lambda)^{-1} \frac{\epsilon^2}{(1+\epsilon)^2} \int_{S^{2n+1}} |\nabla_{\theta_0} w|_{\theta_0}^2 dV_{\theta_0} \\
& \leq C\delta^{-1} + (\delta\Lambda)^{-1} \left(2 + \frac{2}{n}\right)^{-1} \frac{\epsilon^2}{1+2\epsilon} \int_{S^{2n+1}} R_h v^{\frac{2}{n}} w^2 dV_{\theta_0}
\end{aligned}$$



where the last inequality follows from (4.66) with  $\beta = 1 + 2\epsilon$ . If we set  $I = \int_{S^{2n+1}} R_h v^{\frac{2}{n}} w^2 dV_{\theta_0}$ , then

$$\begin{aligned} I &= \int_{\{|R_h| > b\}} |R_h| v^{\frac{2}{n}} w^2 dV_{\theta_0} + \int_{\{|R_h| \leq b\}} |R_h| v^{\frac{2}{n}} w^2 dV_{\theta_0} \\ &\leq a_p^{\frac{1}{p}} \left( \frac{a_p}{b^p} \right)^{\frac{1}{n+1} - \frac{1}{p}} \text{Vol}(S^{2n+1}, \theta_0)^{\frac{1}{n+1}} \left( \int_{S^{2n+1}} w^{2+\frac{2}{n}} dV_{\theta_0} \right)^{\frac{n}{n+1}} + b \int_{S^{2n+1}} v^{\frac{2}{n}} w^2 dV_{\theta_0} \\ &\leq a_p^{\frac{1}{p}} \left( \frac{a_p}{b^p} \right)^{\frac{1}{n+1} - \frac{1}{p}} \text{Vol}(S^{2n+1}, \theta_0)^{\frac{1}{n+1}} \left( \int_{S^{2n+1}} w^{2+\frac{2}{n}} dV_{\theta_0} \right)^{\frac{n}{n+1}} \\ &\quad + bC\delta^{-1} + b(\delta\Lambda)^{-1} \left( 2 + \frac{2}{n} \right)^{-1} \frac{\epsilon^2}{1+2\epsilon} I \end{aligned}$$

where the first inequality follows from (4.69), and the last inequality follows from (4.75). This implies that

$$\begin{aligned} &\left[ 1 - b(\delta\Lambda)^{-1} \left( 2 + \frac{2}{n} \right)^{-1} \frac{\epsilon^2}{1+2\epsilon} \right] I \\ &\leq a_p^{\frac{1}{p}} \left( \frac{a_p}{b^p} \right)^{\frac{1}{n+1} - \frac{1}{p}} \text{Vol}(S^{2n+1}, \theta_0)^{\frac{1}{n+1}} \left( \int_{S^{2n+1}} w^{2+\frac{2}{n}} dV_{\theta_0} \right)^{\frac{n}{n+1}} + bC\delta^{-1}. \end{aligned}$$

First, choose  $b$  large enough such that

$$a_p^{\frac{1}{p}} \left( \frac{a_p}{b^p} \right)^{\frac{1}{n+1} - \frac{1}{p}} < \frac{n(n+1)}{16}.$$

Next, choose  $\epsilon > 0$  small enough such that

$$1 - b(\delta\Lambda)^{-1} \left( 2 + \frac{2}{n} \right)^{-1} \frac{\epsilon^2}{1+2\epsilon} \geq \frac{1}{2}.$$

Thus

$$(4.76) \quad I \leq \frac{n(n+1)}{8} \text{Vol}(S^{2n+1}, \theta_0)^{\frac{1}{n+1}} \left( \int_{S^{2n+1}} w^{2+\frac{2}{n}} dV_{\theta_0} \right)^{\frac{n}{n+1}} + C.$$

Now by our choice of  $\beta = 1 + 2\epsilon$  with  $\epsilon < 1$ , we conclude that  $\frac{(1+\beta)^2}{4\beta} \leq 2$ . This together with (4.67) and (4.76) implies that

$$\begin{aligned} &\frac{n(n+1)}{2} \text{Vol}(S^{2n+1}, \theta_0)^{\frac{1}{n+1}} \left( \int_{S^{2n+1}} w^{2+\frac{2}{n}} dV_{\theta_0} \right)^{\frac{n}{n+1}} \leq C \int_{S^{2n+1}} w^2 dV_{\theta_0} + 2I \\ &\leq C \int_{S^{2n+1}} w^2 dV_{\theta_0} + \frac{n(n+1)}{4} \text{Vol}(S^{2n+1}, \theta_0)^{\frac{1}{n+1}} \left( \int_{S^{2n+1}} w^{2+\frac{2}{n}} dV_{\theta_0} \right)^{\frac{n}{n+1}} + C, \end{aligned}$$

which gives

$$(4.77) \quad \frac{n(n+1)}{4} \text{Vol}(S^{2n+1}, \theta_0)^{\frac{1}{n+1}} \left( \int_{S^{2n+1}} w^{2+\frac{2}{n}} dV_{\theta_0} \right)^{\frac{n}{n+1}} \leq C \int_{S^{2n+1}} w^2 dV_{\theta_0} + C.$$

On the other hand, if  $\epsilon \leq \frac{1}{n}$ , then

$$(4.78) \quad \begin{aligned} \int_{S^{2n+1}} w^2 dV_{\theta_0} &= \int_{S^{2n+1}} v^{2+2\epsilon} dV_{\theta_0} \\ &\leq \left( \int_{S^{2n+1}} v^{2+\frac{2}{n}} dV_{\theta_0} \right)^{\frac{n(1+\epsilon)}{1+n}} \text{Vol}(S^{2n+1}, \theta_0)^{1-\frac{n(1+\epsilon)}{1+n}} \leq C \end{aligned}$$

by Hölder's inequality and assumption (i). It follows from (4.77) and (4.78) that

$$\left( \int_{S^{2n+1}} v^{2+\frac{2}{n}+\epsilon_0} dV_{\theta_0} \right)^{\frac{n}{n+1}} = \left( \int_{S^{2n+1}} w^{2+\frac{2}{n}} dV_{\theta_0} \right)^{\frac{n}{n+1}} \leq C$$

where  $\epsilon_0 = (2 + \frac{2}{n})\epsilon > 0$ . This proves the assertion.  $\square$

**Lemma 4.15.** *Let  $\{\theta_k = \theta(t_k)\}$  be any sequence for the flow with initial data  $u_0 \in C_f^\infty$ . If  $f$  cannot be realized as a Wester scalar curvature in its conformal class, then there exists a subsequence  $\{t_j\}$  such that*

$$v(t_j) \rightarrow 1 \quad \text{in } C_P^{1,\lambda}(S^{2n+1})$$

for some  $\lambda \in (0, 1)$ . Here  $C_P^{1,\lambda}(S^{2n+1})$  is the parabolic Hörmander Hölder spaces defined as in (2.26).

*Proof.* For simplicity, set  $\phi_k = \phi(t_k)$  to be the CR diffeomorphism such that the normalized contact form  $h_k$  is given by  $h_k = h(t_k) = \phi_k^*(\theta(t_k)) = v(t_k)^{\frac{2}{n}} \theta_0$ . By Lemma 2.4 and Lemma 3.2, we have

$$(4.79) \quad \begin{aligned} \|R_{h_k}\|_{L^p(S^{2n+1}, h_k)} &\leq \|\alpha(t_k)f \circ \phi_k - R_{h_k}\|_{L^p(S^{2n+1}, h_k)} + \|\alpha(t_k)f \circ \phi_k\|_{L^p(S^{2n+1}, h_k)} \\ &= \|\alpha(t_k)f - R_{\theta_k}\|_{L^p(S^{2n+1}, \theta_k)} + \|\alpha(t_k)f\|_{L^p(S^{2n+1}, \theta_k)} \leq C \end{aligned}$$

for any  $p \geq 1$ . Note that condition (i) in Lemma 4.14 is satisfied because of (4.34) and Proposition 2.1, condition (ii) is fulfilled by (4.79), and condition (iii) is satisfied thanks to Lemma 4.13. Finally by Lemma 4.12, if we choose  $\sigma_0 = C_3$  and  $l_0 = \text{Vol}(S^{2n+1}, \theta_0)$ , condition (iv) in Lemma 4.14 is fulfilled. Hence, we can apply Lemma 4.14 to  $v(t_k)$  to show that there exists  $\epsilon_0 > 0$  and  $C_0 > 0$  such that

$$(4.80) \quad \int_{S^{2n+1}} v(t_k)^{2+\frac{2}{n}+\epsilon_0} dV_{\theta_0} \leq C_0.$$

For simplicity, set  $v(t_k) = v_k$ . Starting with  $q_0 = 2 + \frac{2}{n} + \epsilon_0$ ,  $p_0 = \frac{q_0}{\frac{1}{2n+2}q_0 + \frac{n+1}{n}} > \frac{2n+2}{n+2}$ ,  $r_0 = \frac{q_0 - (2 + \frac{2}{n})}{(1 + \frac{2}{n})p_0 - (2 + \frac{2}{n})} > 1$ , we have

$$\begin{aligned} \int_{S^{2n+1}} |R_{h_k} v_k^{1+\frac{2}{n}}|^{p_0} dV_{\theta_0} &= \int_{S^{2n+1}} |R_{h_k}|^{p_0} v_k^{(1+\frac{2}{n})p_0} dV_{\theta_0} \\ &\leq \left( \int_{S^{2n+1}} |R_{h_k}|^{\frac{p_0 r_0}{r_0-1}} v_k^{2+\frac{2}{n}} dV_{\theta_0} \right)^{\frac{r_0-1}{r_0}} \left( \int_{S^{2n+1}} v_k^{q_0} dV_{\theta_0} \right)^{\frac{1}{r_0}} \leq C \end{aligned}$$

by Hölder's inequality, (4.79) and (4.80). Hence,

$$(4.81) \quad \left(2 + \frac{2}{n}\right) \Delta_{\theta_0} v = R_{\theta_0} v - R_h v^{1+\frac{2}{n}} \in L^{p_0}(S^{2n+1}, \theta_0).$$

Hence,  $v_k \in S_2^{p_0}(S^{2n+1}, \theta_0)$  by Theorem 3.16 in [18] (see also [20]). By Folland-Stein embedding theorem (see [20] or Theorem 3.13 in [18]), we obtain

$$v_k \in S_2^{p_0}(S^{2n+1}, \theta_0) \hookrightarrow L^{\frac{p_0(n+1)}{n+1-p_0}}(S^{2n+1}, \theta_0) = L^{q_1}(S^{2n+1}, \theta_0)$$

with  $q_1 = \frac{p_0(n+1)}{n+1-p_0}$ . Then we set  $p_1 = \frac{q_1}{\frac{1}{2n+2}q_1 + \frac{n+1}{n}}$ ,  $r_1 = \frac{q_1 - (2 + \frac{2}{n})}{(1 + \frac{2}{n})p_1 - (2 + \frac{2}{n})}$ . In general, if we know  $q_{l-1}, p_{l-1}, r_{l-1}$ , we can define inductively

$$(4.82) \quad \begin{aligned} q_l &= \frac{p_{l-1}(n+1)}{n+1-p_{l-1}} = \frac{2n(n+1)q_{l-1}}{2(n+1)^2 - nq_{l-1}}, \quad p_l = \frac{q_l}{\frac{1}{2n+2}q_l + \frac{n+1}{n}}, \\ r_l &= \frac{q_l - (2 + \frac{2}{n})}{(1 + \frac{2}{n})p_l - (2 + \frac{2}{n})} > 1, \end{aligned}$$

where  $l \in \mathbb{Z}^+$ . Note that if  $2 + \frac{2}{n} < q_l < \frac{2(n+1)^2}{n}$ , then by (4.82)

$$q_{l+1} - q_l = \left( \frac{q_l - (2 + \frac{2}{n})}{\frac{2(n+1)^2}{n} - q_l} \right) q_l \geq \left( \frac{\epsilon_0}{n(2 + \frac{2}{n}) - \epsilon_0} \right) q_l$$

since  $q_l \geq q_0 \geq 2 + \frac{2}{n} + \epsilon_0$ . Thus there exists  $l_0 \in \mathbb{N}$  with  $q_{l_0} > \frac{2(n+1)^2}{n}$  and  $q_0 < q_1 < \dots < q_{l_0-1} < \frac{2(n+1)^2}{n} < q_{l_0}$  such that

$$p_{l_0} = \frac{q_{l_0}}{\frac{1}{2n+2}q_{l_0} + \frac{n+1}{n}} > n+1.$$

Therefore by using the similar argument to get (4.81), we can prove that

$$(2 + \frac{2}{n})\Delta_{\theta_0}v = R_{\theta_0}v - R_h v^{1+\frac{2}{n}} \in L^{p_{l_0}}(S^{2n+1}, \theta_0).$$

Hence, by using Theorem 3.17 in [18] (see also [20]), we can conclude that

$$\|v_k\|_{C_P^\sigma(S^{2n+1})} \leq C \quad \text{with } \sigma = 2 - \frac{2n+2}{p_{l_0}},$$

Consequently, from (4.81) of  $v_j$  and Lemma 3.2, we conclude that

$$v_j \rightarrow v_\infty \quad \text{in } C_P^{1,\lambda}(S^{2n+1}), \text{ for any } \lambda \in (0, 1), \text{ as } j \rightarrow \infty.$$

Recall now that  $\widehat{P(t_k)}$  is the center of mass of the contact form  $\theta_k = \theta(t_k)$ , and by the remark right after Lemma 4.3, (4.27) holds and hence  $\widehat{P(t_k)} \rightarrow Q$  as  $k \rightarrow \infty$  if the sequence is not bounded in  $S_2^p(S^{2n+1}, \theta_0)$  for any  $p > n+1$ . For the normalized conformal CR diffeomorphisms  $\phi_k = \phi(t_k) = \phi_{p(t_k), r(t_k)}$  where  $p(t_k) \in \mathbb{H}^n$  and  $r(t_k) > 0$ , if there exists a subsequence  $\{t_j\}$  such that  $r_j = r(t_j) \rightarrow r_0 < \infty$ , then  $\phi_j \rightarrow \phi_{Q, r_0}$  as  $j \rightarrow \infty$ . Since  $v_j$  is bounded from above and below by positive constants, and so is  $\det(d\phi(t_j)) \rightarrow \det(d\phi_{Q, r_0})$ , we conclude that  $\{u_j = u(t_j)\}$  is uniformly bounded from above and below by positive constants. Hence, there exists a convergent subsequence with  $u_\infty$  as limit. Then by assumption  $R_{\theta(t_k)} \rightarrow R_\infty$  in  $L^p(S^{2n+1}, \theta_0)$  with  $p > n+1$  and on the other hand  $R_{\theta(t_k)} - \alpha(t_k)f \rightarrow 0$  in  $L^p(S^{2n+1}, \theta_0)$  with  $p > n+1$ , hence up to a subsequence  $R_\infty = cf$ . Thus the Webster scalar curvature of the contact form  $u_\infty^{\frac{2}{n}}\theta_0$  is equal to  $f$  up to a constant, which contradicts our assumption that  $f$  cannot be realized as the Webster scalar

curvature in the conformal class. Therefore  $r(t_k) \rightarrow \infty$  as  $k \rightarrow \infty$ . Furthermore, there holds

$$(4.83) \quad \begin{aligned} & \phi_{p(t_k), r(t_k)} \rightarrow \phi_{Q, \infty} \equiv Q \quad \text{uniformly for } x \in S^{2n+1} \setminus B_\delta(Q) \\ & \text{with any sufficiently small } \delta > 0. \end{aligned}$$

Thus, from (4.27), (4.83) and our assumption on  $R_\infty$ , letting  $k \rightarrow \infty$ , we see that

$$\begin{aligned} & \|R_{h_k} - R_\infty(Q)\|_{L^p(S^{2n+1}, h_k)} \\ & \leq \|R_{\theta_k} \circ \phi_k - R_\infty \circ \phi_k\|_{L^p(S^{2n+1}, h_k)} + \|R_\infty \circ \phi_k - R_\infty(Q)\|_{L^p(S^{2n+1}, h_k)} \\ & = \|R_{\theta_k} - R_\infty\|_{L^p(S^{2n+1}, \theta_k)} + \|R_\infty - R_\infty(Q)\|_{L^p(S^{2n+1}, \theta_k)} \rightarrow 0, \end{aligned}$$

for any  $p > n + 1$ . Hence, we have shown that  $v_\infty$  weakly solves

$$-(2 + \frac{2}{n})\Delta_{\theta_0} v_\infty + R_{\theta_0} v_\infty = R_\infty(Q) v_\infty^{1 + \frac{2}{n}} \quad \text{on } S^{2n+1}.$$

Since  $v_k$  satisfies normalization (4.22) and  $\text{Vol}(S^{2n+1}, h_k) = \int_{S^{2n+1}} u_0^{2 + \frac{2}{n}} dV_{\theta_0} = \text{Vol}(S^{2n+1}, \theta_0)$ ,  $v_\infty$  must be constant (see the proof of Theorem 3.1 in [21] or see [29]). Therefore, we asset that  $v_\infty \equiv 1$ , which indicates (4.28). This proves Lemma 4.15  $\square$

**Lemma 4.16.** *Let  $f : S^{2n+1} \rightarrow \mathbb{R}$  be a smooth positive non-degenerate Morse function satisfying the simple bubble condition (sbc):*

$$\frac{\max_{S^{2n+1}} f}{\min_{S^{2n+1}} f} < 2^{\frac{1}{n}}.$$

*Suppose that  $f$  cannot be realized as the Webster scalar curvature of any contact form conformal to  $\theta_0$ . Let  $u(t)$  be a smooth solution of (2.5) with initial data  $u_0 \in C_f^\infty$ . Then there exists a family of CR diffeomorphism  $\phi(t)$  on  $S^{2n+1}$  with the normalized contact form  $h(t) = v(t)^{\frac{2}{n}} \theta_0 = \phi(t)^*(\theta(t))$  such that as  $t \rightarrow \infty$*

$$v(t) \rightarrow 1, \quad h(t) \rightarrow \theta_0 \quad \text{in } C_P^{1, \gamma}(S^{2n+1})$$

*for any  $\gamma \in (0, 1)$ , and  $\phi(t) - \widehat{P}(t) \rightarrow 0$  in  $L^2(S^{2n+1}, \theta_0)$ . Moreover, as  $t \rightarrow \infty$ , we have*

$$\|f \circ \phi(t) - f(\widehat{P}(t))\|_{L^2(S^{2n+1}, \theta_0)} \rightarrow 0 \quad \text{and} \quad \alpha(t) f(\widehat{P}(t)) \rightarrow R_{\theta_0}.$$

*Here  $\widehat{P}(t)$  is defined as in (4.21).*

*Proof.* We prove it by contradiction. Suppose, for a fixed  $\gamma \in (0, 1)$ , there exists a sequence  $t_l \rightarrow \infty$  such that

$$(4.84) \quad \lim_{l \rightarrow \infty} \left( \|v(t_l) - 1\|_{C^{1, \gamma}(S^{2n+1})} + \|\phi(t_l) - \widehat{P}(t_l)\|_{L^2(S^{2n+1}, \theta_0)} \right) > 0.$$

Since Lemma 4.15 implies that

$$(4.85) \quad \lim_{l \rightarrow \infty} \|v(t_l) - 1\|_{C^{1, \gamma}(S^{2n+1})} = 0,$$

by (4.84)

$$\lim_{l \rightarrow \infty} \|\phi(t_l) - \widehat{P}(t_l)\|_{L^2(S^{2n+1}, \theta_0)}^2 \equiv C_0 \text{Vol}(S^{2n+1}, \theta_0)$$

for some constant  $C_0 > 0$ . Observe that  $|\phi(t) - \widehat{P(t)}|^2 = 2(1 - \langle \phi(t), \widehat{P(t)} \rangle)$ . Thus we have

$$\begin{aligned}
 C_0 \text{Vol}(S^{2n+1}, \theta_0) &= \lim_{l \rightarrow \infty} \|\phi(t_l) - \widehat{P(t_l)}\|_{L^2(S^{2n+1}, \theta_0)}^2 \\
 &= \lim_{l \rightarrow \infty} \int_{S^{2n+1}} 2|1 - \langle \phi(t_l), \widehat{P(t)} \rangle| dV_{\theta_0} \\
 (4.86) \quad &= \lim_{l \rightarrow \infty} \int_{S^{2n+1}} 2|1 - \langle \phi(t_l), \widehat{P(t)} \rangle| dV_{h_l} \\
 &\quad + \lim_{l \rightarrow \infty} \int_{S^{2n+1}} 2|1 - \langle \phi(t_l), \widehat{P(t)} \rangle| (dV_{\theta_0} - dV_{h_l}).
 \end{aligned}$$

Clearly, the second limit is zero as it can be seen from (4.85). On the other hand, we have

$$\begin{aligned}
 (4.87) \quad &\int_{S^{2n+1}} 2|1 - \langle \phi(t_l), \widehat{P(t)} \rangle| dV_{h_l} = 2 \left( \text{Vol}(S^{2n+1}, h_l) - \left\langle \int_{S^{2n+1}} \phi(t_l) dV_{h_l}, \widehat{P(t)} \right\rangle \right) \\
 &= 2 \left( \text{Vol}(S^{2n+1}, \theta_0) - \left\langle \int_{S^{2n+1}} x u(t_l)^{2+\frac{2}{n}} dV_{\theta_0}, \widehat{P(t)} \right\rangle \right) \\
 &= 2 \text{Vol}(S^{2n+1}, \theta_0) (1 - \|P(t_l)\|^2).
 \end{aligned}$$

Combining (4.86) and (4.87), we obtain that

$$1 - \|P(t_l)\|^2 \geq \frac{C_0}{2} > 0$$

for  $l$  sufficiently large.

Since  $\|P(t_l)\|^2 \leq 1 - \frac{C_0}{2} < 1$ , there exists a subsequence, still denote as  $\{t_l\}$ , such that  $P(t_l) \rightarrow P_0$  as  $l \rightarrow \infty$  with  $\|P_0\| < 1$ . Then  $\phi(t_l) \rightarrow \phi_{\frac{P_0}{\|P_0\|}, r_0}$  as  $l \rightarrow \infty$  with  $r_0$  being finite. Hence, we can conclude as in the proof of Lemma 4.15 that  $u(t_l)$  is uniformly bounded from above and below by positive constants. Then it implies that  $\{u_l\}$  has a convergent subsequence, and its limit  $u_\infty$  is strictly positive and smooth such that  $u_\infty^{\frac{2}{n}} \theta_0$  has Webster scalar curvature equal to  $f$ , which contradicts the assumption that  $f$  cannot be realized as the Webster scalar curvature in the conformal class.

Moreover, since  $v \rightarrow 1$  in  $C^{1,\gamma}(S^{2n+1})$ , we have

$$\int_{S^{2n+1}} R_h dV_h = \int_{S^{2n+1}} \left( \left(2 + \frac{2}{n}\right) |\nabla_{\theta_0} v|_{\theta_0}^2 + R_{\theta_0} v^2 \right) dV_{\theta_0} \rightarrow R_{\theta_0} \text{Vol}(S^{2n+1}, \theta_0)$$

as  $t \rightarrow \infty$ . Thus, we have

$$\begin{aligned}
 \lim_{t \rightarrow \infty} \left( R_{\theta_0} - \alpha(t) f(\widehat{P(t)}) \right) \cdot \text{Vol}(S^{2n+1}, \theta_0) &= \lim_{t \rightarrow \infty} \left( \int_{S^{2n+1}} R_h dV_h - \alpha(t) f(\widehat{P(t)}) \cdot \text{Vol}(S^{2n+1}, \theta_0) \right) \\
 &= \lim_{t \rightarrow \infty} \alpha(t) \int_{S^{2n+1}} \left( f \circ \phi(t) - f(\widehat{P(t)}) \right) dV_h = 0.
 \end{aligned}$$

Thus the proof of Lemma 4.16 is complete.  $\square$

## 5. SPECTRAL DECOMPOSITION

As before, we denote

$$F_p(t) = \int_{S^{2n+1}} |R_\theta - \alpha f|^p dV_\theta \quad \text{and} \quad G_p(t) = \int_{S^{2n+1}} |\nabla_\theta(R_\theta - \alpha f)|_\theta^p dV_\theta$$

for  $p \geq 1$ .

5.1. Upper bound of change rate of  $F_2(t)$ .

**Lemma 5.1.** *With error  $o(1) \rightarrow 0$  as  $t \rightarrow \infty$ , there holds*

$$\frac{d}{dt} F_2(t) \leq (n+1 + o(1))(nF_2(t) - 2G_2(t)) + o(1)F_2(t).$$

*Proof.* By (3.2), we have

$$\begin{aligned} (5.1) \quad & \frac{d}{dt} \left( \frac{1}{2} \int_{S^{2n+1}} (\alpha f - R_\theta)^2 dV_\theta \right) + \frac{1}{n+1} \left( \frac{\alpha'}{\alpha} \right)^2 E(u) \\ &= \frac{n}{n+1} \frac{\alpha'}{\alpha} \int_{S^{2n+1}} (\alpha f - R_\theta) R_\theta dV_\theta + \int_{S^{2n+1}} (\alpha f - R_\theta)^2 R_\theta dV_\theta \\ &\quad - (n+1) \int_{S^{2n+1}} |\nabla_\theta(\alpha f - R_\theta)|_\theta^2 dV_\theta + \frac{n+1}{2} \int_{S^{2n+1}} (\alpha f - R_\theta)^3 dV_\theta \\ &= \frac{n}{n+1} \frac{\alpha'}{\alpha} \int_{S^{2n+1}} \alpha f (\alpha f - R_\theta) dV_\theta - \frac{n}{n+1} \frac{\alpha'}{\alpha} \int_{S^{2n+1}} (\alpha f - R_\theta)^2 dV_\theta \\ &\quad + \int_{S^{2n+1}} \alpha f (\alpha f - R_\theta)^2 dV_\theta + \frac{n-1}{2} \int_{S^{2n+1}} (\alpha f - R_\theta)^3 dV_\theta - (n+1) \int_{S^{2n+1}} |\nabla_\theta(\alpha f - R_\theta)|_\theta^2 dV_\theta \\ &= \frac{n}{(n+1)E(u)} \left[ -n \int_{S^{2n+1}} (\alpha f - R_\theta)^2 dV_\theta - \int_{S^{2n+1}} \alpha f (\alpha f - R_\theta) dV_\theta \right] \int_{S^{2n+1}} \alpha f (\alpha f - R_\theta) dV_\theta \\ &\quad - \frac{n}{n+1} \frac{\alpha'}{\alpha} \int_{S^{2n+1}} (\alpha f - R_\theta)^2 dV_\theta + \int_{S^{2n+1}} \alpha f (\alpha f - R_\theta)^2 dV_\theta \\ &\quad + \frac{n-1}{2} \int_{S^{2n+1}} (\alpha f - R_\theta)^3 dV_\theta - (n+1) \int_{S^{2n+1}} |\nabla_\theta(\alpha f - R_\theta)|_\theta^2 dV_\theta \end{aligned}$$

where the last equality follows from (2.17). Note that the first term on the right hand side of (5.1) is less than or equals to

$$\begin{aligned} & - \frac{n^2}{(n+1)E(u)} \left( \int_{S^{2n+1}} (\alpha f - R_\theta)^2 dV_\theta \right) \left( \int_{S^{2n+1}} \alpha f (\alpha f - R_\theta) dV_\theta \right) \\ & \leq \frac{n^2}{(n+1)R_{\theta_0} \text{Vol}(S^{2n+1}, \theta_0)} F_2(t) \left( \alpha_2 \left( \max_{S^{2n+1}} f \right) F_2(t)^{\frac{1}{2}} \text{Vol}(S^{2n+1}, \theta_0)^{\frac{1}{2}} \right) = o(1)F_2(t) \end{aligned}$$

by (2.14), Hölder's inequality, Proposition 2.1, Lemma 2.4, and Lemma 3.2. For the second term on the right hand side of (5.1), we can combine it with the second

term on the left hand side to get

$$\begin{aligned}
& \frac{1}{n+1} \left( \frac{\alpha'}{\alpha} \right)^2 E(u) + \frac{n}{n+1} \frac{\alpha'}{\alpha} \int_{S^{2n+1}} (\alpha f - R_\theta)^2 dV_\theta \\
&= \frac{1}{n+1} \left[ \frac{\alpha'}{\alpha} \sqrt{E(u)} + \frac{n \int_{S^{2n+1}} (\alpha f - R_\theta)^2 dV_\theta}{2\sqrt{E(u)}} \right]^2 - \frac{n^2}{4(n+1)E(u)} \left( \int_{S^{2n+1}} (\alpha f - R_\theta)^2 dV_\theta \right)^2 \\
&\geq - \frac{n^2}{4(n+1)R_{\theta_0} \text{Vol}(S^{2n+1}, \theta_0)} \left( \int_{S^{2n+1}} (\alpha f - R_\theta)^2 dV_\theta \right)^2
\end{aligned}$$

by (2.14) and Proposition 2.1. Note also that

$$\begin{aligned}
\left( \int_{S^{2n+1}} (\alpha f - R_\theta)^2 dV_\theta \right)^2 &\leq \text{Vol}(S^{2n+1}, \theta_0)^{\frac{2}{3}} \left( \int_{S^{2n+1}} |\alpha f - R_\theta|^3 dV_\theta \right)^{\frac{4}{3}} \\
&= C \int_{S^{2n+1}} |\alpha f - R_\theta|^3 dV_\theta
\end{aligned}$$

by Hölder's inequality, Proposition 2.1 and Lemma 3.2. Combining all these, we can deduce from (5.1) that

$$(5.2) \quad \frac{1}{2} \frac{d}{dt} F_2(t) \leq \int_{S^{2n+1}} \alpha f (\alpha f - R_\theta)^2 dV_\theta - (n+1) G_2(t) + C \int_{S^{2n+1}} |\alpha f - R_\theta|^3 dV_\theta + o(1) F_2(t).$$

Observe that

$$\begin{aligned}
(5.3) \quad & \int_{S^{2n+1}} |\alpha f - R_\theta|^3 dV_\theta \\
&\leq \left( \int_{S^{2n+1}} |\alpha f - R_\theta|^{n+1} dV_\theta \right)^{\frac{1}{n+1}} \left( \int_{S^{2n+1}} |\alpha f - R_\theta|^{2+\frac{2}{n}} dV_\theta \right)^{\frac{n}{n+1}} \\
&\leq F_{n+1}(t)^{\frac{1}{n+1}} \cdot \frac{1}{Y(S^{2n+1}, \theta_0)} \left( \int_{S^{2n+1}} \left( 2 + \frac{2}{n} \right) |\nabla_\theta (\alpha f - R_\theta)|_\theta^2 + R_\theta |\alpha f - R_\theta|^2 dV_\theta \right) \\
&= o(1) \left( \int_{S^{2n+1}} |\nabla_\theta (\alpha f - R_\theta)|_\theta^2 + |\alpha f - R_\theta|^2 + |\alpha f - R_\theta|^3 dV_\theta \right)
\end{aligned}$$

by Hölder's inequality, Lemma 2.3, 2.4, and 3.2, which implies that

$$(5.4) \quad \int_{S^{2n+1}} |\alpha f - R_\theta|^3 dV_\theta \leq o(1) (G_2(t) + F_2(t)).$$

Finally, we rewrite the first term in (5.2) as

$$\begin{aligned}
(5.5) \quad & \int_{S^{2n+1}} \alpha f (\alpha f - R_\theta)^2 dV_\theta = \int_{S^{2n+1}} \alpha (f_\phi - f(\widehat{P(t)})) (\alpha f_\phi - R_h)^2 dV_h \\
&\quad + \left( \alpha f(\widehat{P(t)}) - \frac{n(n+1)}{2} \right) \int_{S^{2n+1}} (\alpha f_\phi - R_h)^2 dV_h \\
&\quad + \frac{n(n+1)}{2} \int_{S^{2n+1}} (\alpha f_\phi - R_h)^2 dV_h.
\end{aligned}$$

By Lemma 4.16,  $\alpha f(\widehat{P(t)}) \rightarrow R_{\theta_0} = \frac{n(n+1)}{2}$  as  $t \rightarrow \infty$ , we obtain that

$$(5.6) \quad \left( \alpha f(\widehat{P(t)}) - \frac{n(n+1)}{2} \right) \int_{S^{2n+1}} (\alpha f_\phi - R_h)^2 dV_h = o(1) F_2(t).$$

To get control of the first term on the right hand side of (5.5), first we notice that

$$\int_{S^{2n+1}} |f_\phi - f(\widehat{P(t)})|^{n+1} dV_h \leq C \int_{S^{2n+1}} |f_\phi - f(\widehat{P(t)})|^{n+1} dV_{\theta_0} = o(1)$$

by Lemma 4.16. Hence, by (5.3), Hölder's inequality, Lemma 2.3 and 2.4, we have

$$(5.7) \quad \begin{aligned} & \left| \int_{S^{2n+1}} \alpha(f_\phi - f(\widehat{P(t)}))(\alpha f_\phi - R_h)^2 dV_h \right| \\ & \leq \alpha_2 \left( \int_{S^{2n+1}} |f_\phi - f(\widehat{P(t)})|^{n+1} dV_h \right)^{\frac{1}{n+1}} \left( \int_{S^{2n+1}} |\alpha f_\phi - R_h|^{2+\frac{2}{n}} dV_h \right)^{\frac{n}{n+1}} \\ & \leq o(1) \left( \int_{S^{2n+1}} (2 + \frac{2}{n}) |\nabla_\theta(\alpha f - R_\theta)|_\theta^2 + R_\theta |\alpha f - R_\theta|^2 dV_\theta \right) \\ & \leq o(1)(G_2(t) + F_2(t)). \end{aligned}$$

Substituting (5.3)-(5.7) into (5.2), we deduce

$$\frac{1}{2} \frac{d}{dt} F_2(t) \leq \frac{n(n+1)}{2} F_2(t) - (n+1)G_2(t) + o(1)(G_2(t) + F_2(t)),$$

as required.  $\square$

**5.2. The spectral decomposition.** Let  $\{\varphi_i\}$  be an  $L^2(S^{2n+1}, \theta_0)$ -orthonormal basis of eigenfunctions of  $-\Delta_{\theta_0}$ , satisfying  $-\Delta_{\theta_0} \varphi_i = \lambda_i \varphi_i$  with eigenvalues  $0 = \lambda_0 < \lambda_1 = \dots = \lambda_{2n+2} = \frac{n}{2} < \lambda_{2n+3} \leq \dots$ . Now in terms of the orthonormal basis  $\{\varphi_i^\theta\}, \{\varphi_i^h\}$  of the eigenfunctions of  $-\Delta_\theta, -\Delta_h$  with the corresponding eigenvalues  $\lambda_i^\theta, \lambda_i^h$  respectively, we expand

$$(5.8) \quad \alpha f - R_\theta = \sum_{i=0}^{\infty} \beta_\theta^i \varphi_i^\theta \quad \text{and} \quad \alpha f_\phi - R_h = \sum_{i=0}^{\infty} \beta_h^i \varphi_i^h,$$

with coefficients

$$(5.9) \quad \beta_h^i = \int_{S^{2n+1}} (\alpha f_\phi - R_h) \varphi_i^h dV_h = \int_{S^{2n+1}} (\alpha f - R_\theta) \varphi_i^\theta dV_\theta = \beta_\theta^i$$

for all  $i \in \mathbb{N}$ . First notice that we always have

$$(5.10) \quad \beta_\theta^0 = 0$$

in view of (2.4). It is well known that  $\varphi_i^h = \varphi_i^\theta \circ \phi$ , which implies (5.9) and  $\lambda_i^\theta = \lambda_i^h$  for all  $i \in \mathbb{N}$ .

**Lemma 5.2.** *As  $t \rightarrow \infty$ , we have  $\lambda_i^\theta = \lambda_i^h \rightarrow \lambda_i$  and we can choose  $\varphi_i$  such that  $\varphi_i^h \rightarrow \varphi_i$  in  $L^2(S^{2n+1}, \theta_0)$  for all  $i \in \mathbb{N}$ .*

*Proof.* First, for  $i = 0$ , for any time  $t$ , the eigenvalue  $\lambda_0^h = \lambda_0 = 0$  and  $\varphi_0^h = \varphi_0 = 1$ . Thus the statement is true for  $i = 0$ . Now assume that the statement is true for all  $i \leq k$  for a fixed integer  $k \geq 0$ . We should show that it is true for  $i = k + 1$  for a suitable choice of  $\varphi_{k+1}$ . As starting point, we show that  $\lambda_{k+1}^h \rightarrow \lambda_{k+1}$ . The argument in the proof of Lemma 4.13 shows that  $\liminf_{t \rightarrow \infty} \lambda_{k+1}^h \geq$



$\lambda_{k+1}$ . Hence we only need to show that  $\limsup_{t \rightarrow \infty} \lambda_{k+1}^h \leq \lambda_{k+1}$ . To do this, we pick  $\varphi_{k+1}$  the eigenfunction for  $-\Delta_{\theta_0}$  which is perpendicular to all  $\varphi_j$  for  $0 \leq j \leq k$  with the smaller eigenvalue. Since by assumption  $\varphi_j^h \rightarrow \varphi_j$  as  $t \rightarrow \infty$  and  $\int_{S^{2n+1}} \varphi_{k+1} \varphi_j dV_{\theta_0} = 0$  for all  $0 \leq j \leq k$ , we conclude that

$$(5.11) \quad \int_{S^{2n+1}} \varphi_{k+1} \varphi_j^h dV_{\theta_0} = o(1) \quad \text{as } t \rightarrow \infty.$$

Now consider the test function

$$(5.12) \quad \Psi^h = \varphi_{k+1} - \sum_{j=0}^k \left( \int_{S^{2n+1}} \varphi_{k+1} \varphi_j^h dV_h \right) \varphi_j^h.$$

Since  $\{\varphi_i^h\}_{i \in \mathbb{N}}$  are orthonormal, one has

$$(5.13) \quad \begin{aligned} \int_{S^{2n+1}} \Psi^h \varphi_l^h dV_h &= \int_{S^{2n+1}} \varphi_{k+1} \varphi_l^h dV_h - \sum_{j=0}^k \left( \int_{S^{2n+1}} \varphi_{k+1} \varphi_j^h dV_h \right) \int_{S^{2n+1}} \varphi_j^h \varphi_l^h dV_h \\ &= \int_{S^{2n+1}} \varphi_{k+1} \varphi_l^h dV_h - \sum_{j=0}^k \left( \int_{S^{2n+1}} \varphi_{k+1} \varphi_j^h dV_h \right) \delta_{jl} = 0 \quad \text{for all } 0 \leq l \leq k. \end{aligned}$$

Thus by characterization of the  $(k+1)$ -th eigenvalue, one obtains

$$(5.14) \quad \lambda_{k+1}^h \leq \frac{\int_{S^{2n+1}} |\nabla_h \Psi^h|_h^2 dV_h}{\int_{S^{2n+1}} (\Psi^h)^2 dV_h}.$$

Note that

$$\begin{aligned} \int_{S^{2n+1}} (\Psi^h)^2 dV_h &= \int_{S^{2n+1}} \Psi^h \varphi_{k+1} dV_h - \sum_{j=0}^k \left( \int_{S^{2n+1}} \varphi_{k+1} \varphi_j^h dV_h \right) \int_{S^{2n+1}} \Psi^h \varphi_j^h dV_h \\ &= \int_{S^{2n+1}} \Psi^h \varphi_{k+1} dV_h \\ &= \int_{S^{2n+1}} \varphi_{k+1}^2 dV_h - \sum_{j=0}^k \left( \int_{S^{2n+1}} \varphi_{k+1} \varphi_j^h dV_h \right)^2 = 1 + o(1) \quad \text{as } t \rightarrow \infty, \end{aligned}$$

where the second equality follows from (5.13), and the last equality follows from (4.28) and (5.11). Note also that

$$\begin{aligned} \int_{S^{2n+1}} |\nabla_h \Psi^h|_h^2 dV_h &= \int_{S^{2n+1}} |\nabla_h \varphi_{k+1}|_h^2 dV_h - 2 \sum_{j=0}^k \left( \int_{S^{2n+1}} \varphi_{k+1} \varphi_j^h dV_h \right) \int_{S^{2n+1}} \langle \nabla_h \varphi_{k+1}, \nabla_h \varphi_j^h \rangle_h dV_h \\ &\quad + \sum_{j,l=0}^k \left( \int_{S^{2n+1}} \varphi_{k+1} \varphi_j^h dV_h \right) \left( \int_{S^{2n+1}} \varphi_{k+1} \varphi_l^h dV_h \right) \int_{S^{2n+1}} \langle \nabla_h \varphi_j^h, \nabla_h \varphi_l^h \rangle_h dV_h \\ &= \lambda_{k+1} + o(1) \quad \text{as } t \rightarrow \infty, \end{aligned}$$

where we have used (4.28) and (5.11). Now if we set  $\varphi_{k+1}^h = \Psi^h / \left( \int_{S^{2n+1}} (\Psi^h)^2 dV_h \right)^{\frac{1}{2}}$ , then the required estimate follows easily from (5.14) and these two estimates.

Since  $\{\varphi_{k+1}^h\}$  is a bounded sequence in  $S_2^2(S^{2n+1}, \theta_0)$ , there exists a subsequence of time  $t_l$  such that  $\{\varphi_{k+1}^h(t_l)\}$  weakly converges to  $\varphi_{k+1}^0$ . Hence it will converge to  $\varphi_{k+1}^0$  strongly in  $S_1^2(S^{2n+1}, \theta_0)$ . In particular, we have

$$\int_{S^{2n+1}} (\varphi_{k+1}^0)^2 dV_{\theta_0} = 1 \quad \text{and} \quad \int_{S^{2n+1}} |\nabla_{\theta_0} \varphi_{k+1}^0|_{\theta_0}^2 dV_{\theta_0} = \lambda_{k+1}.$$

Then  $\lim_{k \rightarrow \infty} \int_{S^{2n+1}} |\nabla_{\theta_0} \varphi_{k+1}^h|_{\theta_0}^2 dV_{\theta_0} = \lambda_{k+1}$ , and  $\varphi_{k+1}^0$  is in the eigenfunction of the eigenvalue  $\lambda_{k+1}$  and is orthogonal to all eigenfunctions  $\varphi_j$  with  $0 \leq j \leq k$ . Redefine  $\varphi_{k+1}$  to be  $\varphi_{k+1}^0$  to finish the induction argument.  $\square$

**5.3. Convergence of CR Yamabe flow.** Now we consider the case when  $f$  is a positive constant function, i.e.  $f \equiv c$  for some positive constant  $c > 0$ . Then by (2.4), we have

$$\alpha f = \frac{\alpha \int_{S^{2n+1}} f dV_{\theta}}{\int_{S^{2n+1}} dV_{\theta}} = \frac{\int_{S^{2n+1}} R_{\theta} dV_{\theta}}{\int_{S^{2n+1}} dV_{\theta}} = \overline{R}_{\theta}$$

where  $\overline{R}_{\theta}$  is the average of the Webster scalar curvature  $R_{\theta}$  of  $\theta$ . Therefore, the Webster scalar curvature flow in (2.3) reduced to the CR Yamabe flow

$$\frac{\partial}{\partial t} \theta = -(R_{\theta} - \overline{R}_{\theta}) \theta, \quad \theta|_{t=0} = u_0^{\frac{2}{n}} \theta_0.$$

Our aim is to prove the following result, which recovers the result of the author in [28]:

**Theorem 5.3.** *Suppose that  $u_0$  satisfies (2.2) and  $f$  is a positive constant function. Then the flow (2.3) converges to a contact form  $\theta_{\infty} = u_{\infty}^{\frac{2}{n}} \theta_0$  of constant Webster scalar curvature.*

*Proof.* First we show the exponential decay of  $F_2$ . We define

$$b = (b^1, \dots, b^{2n+2}) = \int_{S^{2n+1}} (x, \overline{x})(\overline{R}_h - R_h) dV_h$$

where  $x = (x_1, \dots, x_{n+1}) \in S^{2n+1} \subset \mathbb{C}^{n+1}$  and  $\overline{x} = (\overline{x}_1, \dots, \overline{x}_{n+1})$ . That is,

$$b^i = \int_{S^{2n+1}} x_i (\overline{R}_h - R_h) dV_h \quad \text{and} \quad b^{n+1+i} = \int_{S^{2n+1}} \overline{x}_i (\overline{R}_h - R_h) dV_h \quad \text{for } 1 \leq i \leq n+1. \quad (5.15)$$

J. H. Cheng proved the following Kazdan-Warner type condition in [14]:

$$(5.16) \quad \int_{S^{2n+1}} \langle \nabla_{\theta_0} x_i, \nabla_{\theta_0} R_h \rangle_{\theta_0} dV_h = 0 \quad \text{and} \quad \int_{S^{2n+1}} \langle \nabla_{\theta_0} \overline{x}_i, \nabla_{\theta_0} R_h \rangle_{\theta_0} dV_h = 0$$

for  $1 \leq i \leq n+1$ . Since

$$-\Delta_{\theta_0} x_i = \frac{n}{2} x_i \quad \text{and} \quad -\Delta_{\theta_0} \overline{x}_i = \frac{n}{2} \overline{x}_i \quad \text{for } 1 \leq i \leq n+1,$$

by (5.16) and integration by parts we find that

$$(5.17) \quad \begin{aligned} b^i &= \int_{S^{2n+1}} (\overline{R}_h - R_h) x_i dV_h = -\frac{2}{n} \int_{S^{2n+1}} (\overline{R}_h - R_h) \Delta_{\theta_0} x_i dV_h \\ &= \frac{4n+4}{n^2} \int_{S^{2n+1}} (\overline{R}_h - R_h) \langle \nabla_{\theta_0} x_i, \nabla_{\theta_0} v \rangle_{\theta_0} v^{1+\frac{2}{n}} dV_{\theta_0} \quad \text{for } 1 \leq i \leq n+1. \end{aligned}$$

Similarly, we have

$$(5.18) \quad b^{n+1+i} = \frac{4n+4}{n^2} \int_{S^{2n+1}} (\bar{R}_h - R_h) \langle \nabla_{\theta_0} \bar{x}_i, \nabla_{\theta_0} v \rangle_{\theta_0} v^{1+\frac{2}{n}} dV_{\theta_0} \text{ for } 1 \leq i \leq n+1.$$

It follows from (5.17), (5.18) and Lemma 4.16 that with error  $o(1) \rightarrow 0$  as  $t \rightarrow \infty$

$$(5.19) \quad |b^i| \leq CF_2(t)^{\frac{1}{2}} \|v - 1\|_{S^2_1(S^{2n+1}, \theta_0)} = o(1)F_2(t)^{\frac{1}{2}} \text{ for } 1 \leq i \leq 2n+2.$$

We can take  $\varphi_i = x_i/\sqrt{n+1}$  and  $\varphi_{n+i} = \bar{x}_i/\sqrt{n+1}$  for  $1 \leq i \leq n+1$ . Hence, by (5.9), (5.15), and Lemma 5.2, we obtain

$$\beta_h^i = \int_{S^{2n+1}} \varphi_i^h (\bar{R}_h - R_h) dV_h = \int_{S^{2n+1}} \varphi_i (\bar{R}_h - R_h) dV_h + o(1) = \frac{b_i}{\sqrt{n+1}} + o(1) \text{ for } 1 \leq i \leq 2n+2,$$

which implies by (5.9) and (5.19) that

$$(5.20) \quad |\beta_\theta^i| = |\beta_h^i| \leq o(1)F_2(t)^{\frac{1}{2}} + o(1) \text{ for } 1 \leq i \leq 2n+2.$$

On the other hand, by (5.8), (5.9) and (5.10), we have

$$(5.21) \quad F_2(t) = \int_{S^{2n+1}} (\bar{R}_\theta - R_\theta)^2 dV_\theta = \sum_{i,j=1}^{\infty} \beta_\theta^i \beta_\theta^j \int_{S^{2n+1}} \varphi_i^\theta \varphi_j^\theta dV_\theta = \sum_{i=1}^{\infty} |\beta_\theta^i|^2$$

and

$$(5.22) \quad \begin{aligned} G_2(t) &= \int_{S^{2n+1}} |\nabla_\theta (\bar{R}_\theta - R_\theta)|_\theta^2 dV_\theta = - \int_{S^{2n+1}} (\bar{R}_\theta - R_\theta) \Delta_\theta (\bar{R}_\theta - R_\theta) dV_\theta \\ &= \sum_{i,j=1}^{\infty} \beta_\theta^i \beta_\theta^j \int_{S^{2n+1}} \varphi_i^\theta (-\Delta_\theta \varphi_j^\theta) dV_\theta \\ &= \sum_{i,j=1}^{\infty} \beta_\theta^i \beta_\theta^j \lambda_j^\theta \int_{S^{2n+1}} \varphi_i^\theta \varphi_j^\theta dV_\theta = \sum_{i=1}^{\infty} \lambda_i^\theta |\beta_\theta^i|^2. \end{aligned}$$

Combining (5.20), (5.21), (5.22) and Lemma 5.2, we obtain

$$(5.23) \quad G_2(t) = \sum_{i=1}^{\infty} \lambda_i^\theta |\beta_\theta^i|^2 \geq (\lambda_{2n+3}^\theta + o(1)) \sum_{i=1}^{\infty} |\beta_\theta^i|^2 = (\lambda_{2n+3} + o(1)) F_2(t).$$

Combining (5.23) and Lemma 5.1 and taking advantage of the spectral gap  $\lambda_{2n+3} > n/2$ , for sufficiently large  $t$  we infer the estimate

$$\frac{d}{dt} F_2(t) \leq -\delta F_2(t)$$

for some uniform constant  $\delta > 0$ , which implies that

$$(5.24) \quad F_2(t) \leq Ce^{-\delta t}$$

for all  $t \geq 0$  with some uniform constant  $C$ .

Note that (5.24) rules out the concentration of volume. Indeed, let  $Q$  be the unique concentration point described in Theorem 4.7, and  $B_{r_0}(Q) = B_{r_0}(Q, \theta_0)$ .

For any  $r_0 > 0$ , we have

(5.25)

$$\begin{aligned} \left| \frac{d}{dt} \text{Vol}(B_{r_0}(Q), \theta) \right| &= \left| \frac{d}{dt} \left( \int_{B_{r_0}(Q)} dV_\theta \right) \right| = (n+1) \left| \int_{B_{r_0}(Q)} (\alpha f - R_\theta) dV_\theta \right| \\ &\leq (n+1) \text{Vol}(S^{2n+1}, \theta)^{\frac{1}{2}} \left( \int_{B_{r_0}(Q)} (\alpha f - R_\theta)^2 dV_\theta \right)^{\frac{1}{2}} \\ &\leq (n+1) \text{Vol}(S^{2n+1}, \theta_0)^{\frac{1}{2}} F_2(t)^{\frac{1}{2}} \leq C e^{-\frac{\delta}{2}t} \quad \text{for all } t \geq 0. \end{aligned}$$

Here we have used (2.5), (5.24) and Hölder's inequality. Thus, by integrating (5.25) from  $T$  to  $t$ , we obtain

$$(5.26) \quad \text{Vol}(B_{r_0}(Q), \theta(t)) \leq \text{Vol}(B_{r_0}(Q), \theta(T)) + \frac{C}{\delta} e^{-\frac{\delta}{2}T}$$

for all  $t \geq T$ . Now, first by choosing  $T$  large enough and then by choosing  $r_0 > 0$  small enough, we obtain

$$\text{Vol}(B_{r_0}(Q), \theta(t)) \leq \frac{1}{2} \text{Vol}(S^{2n+1}, \theta_0) \quad \text{for all } t \geq T,$$

thanks to (5.26). In particular, the concentration in the sense of (4.27) cannot occur. By Theorem 4.7, case (i) occurs. Therefore, Lemma 4.8 implies that  $u(t)$  converges to  $u_\infty$  in  $S_2^p(S^{2n+1}, \theta_0)$  for all  $p > n+1$ , and hence in  $C^\infty(S^{2n+1})$ , as  $t \rightarrow \infty$  such that the Webster scalar curvature of  $\theta_\infty = u_\infty^{\frac{2}{n}} \theta_0$  is constant.  $\square$

## 6. APPENDIX

We have the following lemma which is an improved version of Lemma 2.3:

**Lemma 6.1.** *For any  $\epsilon > 0$ , there exists a constant  $C_\epsilon$  such that for any  $0 \leq u \in S_1^2(S^{2n+1}, \theta_0)$  satisfying*

$$(6.1) \quad \int_{S^{2n+1}} x_i u^{2+\frac{2}{n}} dV_{\theta_0} = \int_{S^{2n+1}} \bar{x}_i u^{2+\frac{2}{n}} dV_{\theta_0} = 0 \quad \text{for all } i = 1, 2, \dots, n+1,$$

where  $x = (x_1, \dots, x_{n+1}) \in S^{2n+1} \subset \mathbb{C}^{n+1}$ , we have

$$(6.2) \quad Y(S^{2n+1}, \theta_0) \left( \int_{S^{2n+1}} u^{2+\frac{2}{n}} dV_{\theta_0} \right)^{\frac{n}{n+1}} \leq \left( 2^{-\frac{1}{n+1}} \cdot \frac{2n+2}{n} + \epsilon \right) \int_{S^{2n+1}} |\nabla_{\theta_0} u|_{\theta_0}^2 dV_{\theta_0} + C_\epsilon \int_{S^{2n+1}} u^2 dV_{\theta_0}.$$

*Proof.* The proof is based on the argument of Aubin in [2] (see also [3]).

Dividing both sides by  $R_{\theta_0} = n(n+1)/2$ , (6.2) is equivalent to

$$(6.3) \quad \text{Vol}(S^{2n+1}, \theta_0)^{\frac{1}{n+1}} \left( \int_{S^{2n+1}} u^{2+\frac{2}{n}} dV_{\theta_0} \right)^{\frac{n}{n+1}} \leq \left( 2^{-\frac{1}{n+1}} \cdot \frac{4}{n^2} + \epsilon \right) \int_{S^{2n+1}} |\nabla_{\theta_0} u|_{\theta_0}^2 dV_{\theta_0} + C_\epsilon \int_{S^{2n+1}} u^2 dV_{\theta_0}.$$

Let  $\Lambda$  be the vector space of functions on  $S^{2n+1}$  spanned by  $x_i$  and  $\bar{x}_i$  where  $i = 1, 2, \dots, n+1$ . Let  $0 < \eta < 1/2$  be a real number, which we are going to choose very small. There exists a family of functions  $\xi_i \in \Lambda$  ( $i = 1, 2, \dots, k$ ) such that

$$(6.4) \quad 1 + \eta < \sum_{i=1}^k |\xi_i|^{\frac{n}{n+1}} < 1 + 2\eta \quad \text{with} \quad |\xi_i| < 2^{-\frac{2n+2}{n}}.$$

Consider  $h_i, C^1$  functions, such that everywhere  $h_i \xi_i \geq 0$  and such that

$$(6.5) \quad ||h_i|^2 - |\xi_i|^{\frac{n}{n+1}}| < (\eta/k)^{\frac{2n+2}{n}}.$$

Then we have

$$(6.6) \quad \begin{aligned} \sum_{i=1}^k |h_i|^2 &< \sum_{i=1}^k |\xi_i|^{\frac{n}{n+1}} + k(\eta/k)^{\frac{n}{2n+2}} < 1 + 2\eta + \eta = 1 + 3\eta \quad \text{and} \\ \sum_{i=1}^k |h_i|^2 &> \sum_{i=1}^k |\xi_i|^{\frac{n}{n+1}} - k(\eta/k)^{\frac{n}{2n+2}} > 1 + \eta - \eta = 1. \end{aligned}$$

By mean-value theorem, we have  $|F(a) - F(b)| \leq \max_{x \in [a, b]} |F'(x)|(a - b)$  for  $a > b$ . Apply this with  $F(x) = x^{\frac{n+1}{n}}$  and note that for  $x$  lies between  $|h_i|^2$  and  $|\xi_i|^{\frac{n}{n+1}}$  satisfying  $|x| = |h_i|^2 + ||h_i|^2 - |\xi_i|^{\frac{n}{n+1}}| < |h_i|^2 + (\eta/k)^{\frac{2n+2}{n}}$  by (6.5), we have

$$(6.7) \quad \begin{aligned} ||h_i|^{\frac{2n+2}{n}} - |\xi_i|| &\leq \frac{n+1}{n} \left[ |\xi_i|^{\frac{n}{n+1}} + (\eta/k)^{\frac{2n+2}{n}} \right]^{\frac{n+1}{n}-1} ||h_i|^2 - |\xi_i|^{\frac{n}{n+1}}| \\ &\leq \frac{n+1}{n} ||h_i|^2 - |\xi_i|^{\frac{n}{n+1}}| \\ &\leq \frac{n+1}{n} (\eta/k)^{\frac{2n+2}{n}} := \epsilon_0^{\frac{2n+2}{n}} \end{aligned}$$

where the second inequality follows from

$$|\xi_i|^{\frac{n}{n+1}} + (\eta/k)^{\frac{2n+2}{n}} < 2^{-2} + (1/2k)^{\frac{2n+2}{n}} < 1$$

by (6.4) and the assumption that  $\eta < 1/2$ . Since  $u \geq 0$ , we have

$$(6.8) \quad \begin{aligned} \|u\|_{L^{\frac{2n+2}{n}}(S^{2n+1}, \theta_0)}^2 &= \|u^2\|_{L^{\frac{n+1}{n}}(S^{2n+1}, \theta_0)} \\ &\leq \left\| \sum_{i=1}^k u^2 |h_i|^2 \right\|_{L^{\frac{n+1}{n}}(S^{2n+1}, \theta_0)} \leq \sum_{i=1}^k \|u h_i\|_{L^{\frac{n+1}{n}}(S^{2n+1}, \theta_0)}^2. \end{aligned}$$

For a function  $f$ , define  $f^+(x) = \sup\{f(x), 0\}$  and  $f^-(x) = \sup\{-f(x), 0\}$ . By assumption (6.1), we have

$$(6.9) \quad \int_{S^{2n+1}} \xi_i^+ u^{\frac{2+2n}{n}} dV_{\theta_0} = \int_{S^{2n+1}} \xi_i^- u^{\frac{2+2n}{n}} dV_{\theta_0}.$$

If

$$(6.10) \quad \|h_i^+ |\nabla_{\theta_0} u|_{\theta_0}\|_{L^2(S^{2n+1}, \theta_0)} \geq \|h_i^- |\nabla_{\theta_0} u|_{\theta_0}\|_{L^2(S^{2n+1}, \theta_0)},$$

we obtain

$$\begin{aligned}
(6.11) \quad & \|uh_i\|_{L^{\frac{2n+2}{n}}(S^{2n+1}, \theta_0)}^2 \\
&= \left[ \int_{S^{2n+1}} (h_i^+ u)^{\frac{2+2n}{n}} dV_{\theta_0} + \int_{S^{2n+1}} (h_i^- u)^{\frac{2+2n}{n}} dV_{\theta_0} \right]^{\frac{n}{n+1}} \\
&\leq \left[ \int_{S^{2n+1}} (\xi_i^+ + \epsilon_0^{\frac{2+2n}{n}}) u^{\frac{2+2n}{n}} dV_{\theta_0} + \int_{S^{2n+1}} (h_i^- u)^{\frac{2+2n}{n}} dV_{\theta_0} \right]^{\frac{n}{n+1}} \\
&= \left[ \int_{S^{2n+1}} (\xi_i^- + \epsilon_0^{\frac{2+2n}{n}}) u^{\frac{2+2n}{n}} dV_{\theta_0} + \int_{S^{2n+1}} (h_i^- u)^{\frac{2+2n}{n}} dV_{\theta_0} \right]^{\frac{n}{n+1}} \\
&\leq 2^{\frac{n}{n+1}} \left[ \int_{S^{2n+1}} ((h_i^-)^{\frac{2+2n}{n}} + \epsilon_0^{\frac{2+2n}{n}}) u^{\frac{2+2n}{n}} dV_{\theta_0} \right]^{\frac{n}{n+1}} \\
&\leq 2^{\frac{n}{n+1}} \|u(h_i^- + \epsilon_0)\|_{L^{\frac{2n+2}{n}}(S^{2n+1}, \theta_0)}^2 \\
&\leq \frac{2^{\frac{n}{n+1}}}{\text{Vol}(S^{2n+1}, \theta_0)^{\frac{1}{n+1}}} \left( \frac{4}{n^2} \|\nabla_{\theta_0}(u(h_i^- + \epsilon_0))\|_{L^2(S^{2n+1}, \theta_0)}^2 + \|u(h_i^- + \epsilon_0)\|_{L^2(S^{2n+1}, \theta_0)}^2 \right)
\end{aligned}$$

where the second and third inequality follows from (6.7), the second equality follows from (6.9), and the last inequality follows from Lemma 2.3. Let  $H = \sup_{1 \leq i \leq k} \sup_{S^{2n+1}} |\nabla_{\theta_0} h_i|_{\theta_0}$ . Then there exist constants  $\mu$  and  $\nu$  such that

$$\begin{aligned}
(6.12) \quad & \|\nabla_{\theta_0}(u(h_i^- + \epsilon_0))\|_{L^2(S^{2n+1}, \theta_0)}^2 \leq \int_{S^{2n+1}} (h_i^- + \epsilon_0)^2 |\nabla_{\theta_0} u|_{\theta_0}^2 dV_{\theta_0} + \nu H^2 \|u\|_{L^2(S^{2n+1}, \theta_0)}^2 \\
& \quad + \mu H \|u\|_{L^2(S^{2n+1}, \theta_0)} \|\nabla_{\theta_0} u\|_{L^2(S^{2n+1}, \theta_0)}.
\end{aligned}$$

Since  $(h_i^- + \epsilon_0)^2 \leq (h_i^-)^2 + 2(h_i^- + \epsilon_0)\epsilon_0 < (h_i^-)^2 + 2\epsilon_0$ , then by (6.8), (6.10), (6.11) and (6.12) we have

$$\begin{aligned}
(6.13) \quad & \text{Vol}(S^{2n+1}, \theta_0)^{\frac{1}{n+1}} \|u\|_{L^{\frac{2n+2}{n}}(S^{2n+1}, \theta_0)}^2 \\
&\leq \frac{4 \cdot 2^{\frac{n}{n+1}}}{n^2} \left( \sum_{i=1}^k \int_{S^{2n+1}} (h_i^- + \epsilon_0)^2 |\nabla_{\theta_0} u|_{\theta_0}^2 dV_{\theta_0} + \nu k H^2 \|u\|_{L^2(S^{2n+1}, \theta_0)}^2 \right. \\
&\quad \left. + \mu k H \|u\|_{L^2(S^{2n+1}, \theta_0)} \|\nabla_{\theta_0} u\|_{L^2(S^{2n+1}, \theta_0)} \right) + 2^{\frac{n}{n+1}} \sum_{i=1}^k \|u(h_i^- + \epsilon_0)\|_{L^2(S^{2n+1}, \theta_0)}^2 \\
&\leq \frac{4 \cdot 2^{\frac{n}{n+1}}}{n^2} \left( \frac{1}{2} \sum_{i=1}^k \int_{S^{2n+1}} h_i^2 |\nabla_{\theta_0} u|_{\theta_0}^2 dV_{\theta_0} + 2k\epsilon_0 \|\nabla_{\theta_0} u\|_{L^2(S^{2n+1}, \theta_0)}^2 \right. \\
&\quad \left. + \nu k H^2 \|u\|_{L^2(S^{2n+1}, \theta_0)}^2 + \mu k H \|u\|_{L^2(S^{2n+1}, \theta_0)} \|\nabla_{\theta_0} u\|_{L^2(S^{2n+1}, \theta_0)} \right) + 2^{\frac{n}{n+1}} k \|u\|_{L^2(S^{2n+1}, \theta_0)}^2.
\end{aligned}$$

Note that if  $\|h_i^+ |\nabla_{\theta_0} u|_{\theta_0}\|_{L^2(S^{2n+1}, \theta_0)} \leq \|h_i^- |\nabla_{\theta_0} u|_{\theta_0}\|_{L^2(S^{2n+1}, \theta_0)}$ , by the same argument above we also have (6.13).

Young's inequality asserts that for any  $\epsilon_1 > 0$ , there exists a constant  $C(\epsilon_1)$  such that

$$(6.14) \quad \|\nabla_{\theta_0} u\|_{L^2(S^{2n+1}, \theta_0)} \|u\|_{L^2(S^{2n+1}, \theta_0)} \leq \epsilon_1 \|\nabla_{\theta_0} u\|_{L^2(S^{2n+1}, \theta_0)}^2 + C(\epsilon_1) \|u\|_{L^2(S^{2n+1}, \theta_0)}^2.$$

Using (6.6) and (6.14), (6.13) can be written as

$$\begin{aligned} \text{Vol}(S^{2n+1}, \theta_0)^{\frac{1}{n+1}} \|u\|_{L^{\frac{2n+2}{n}}(S^{2n+1}, \theta_0)}^2 &\leq \frac{4}{n^2} \left[ 2^{-\frac{1}{n+1}} (1 + 3\eta) + 2^{\frac{2n+1}{n+1}} k \epsilon_0 + \mu k H \epsilon_1 \right] \|\nabla_{\theta_0} u\|_{L^2(S^{2n+1}, \theta_0)}^2 \\ &\quad + 2^{\frac{n}{n+1}} k \left[ \frac{4}{n^2} (\nu H^2 + \mu H C(\epsilon_1)) + 1 \right] \|u\|_{L^2(S^{2n+1}, \theta_0)}^2. \end{aligned}$$

Recall by (6.7),  $\epsilon_0^{\frac{2n+2}{n}} = \frac{n+1}{n} (\eta/k)^{\frac{2n+2}{n}}$ . Taking  $\eta$  and  $\epsilon_1$  small enough, we prove (6.3) and hence (6.2).  $\square$

## REFERENCES

1. T. Aubin, Équations différentielles non linéaires et problème de Yamabe concernant la courbure scalaire. *J. Math. Pures Appl. (9)* **55** (1976), 269–296.
2. T. Aubin, Meilleures constantes dans le théorème d'inclusion de Sobolev et un théorème de Fredholm non linéaire pour la transformation conforme de la courbure scalaire. *J. Funct. Anal.* **32** (1979), 148–174.
3. T. Aubin, Some nonlinear problems in Riemannian geometry. Springer Monographs in Mathematics. Springer, Berlin, (1998).
4. M. Bramanti and L. Brandolini, Schauder estimates for parabolic nondivergence operators of Hörmander type. *J. Differential Equations* **234** (2007), 177–245.
5. S. Brendle, Convergence of the Yamabe flow for arbitrary initial energy. *J. Differential Geom.* **69** (2005), 217–278.
6. S. Brendle, Convergence of the Yamabe flow in dimension 6 and higher. *Invent. Math.* **170** (2007), 541–576.
7. S. C. Chang, H. L. Chiu, and C. T. Wu, The Li-Yau-Hamilton inequality for Yamabe flow on a closed CR 3-manifold. *Trans. Amer. Math. Soc.* **362** (2010), 1681–1698.
8. S. C. Chang and J. H. Cheng, The Harnack estimate for the Yamabe flow on CR manifolds of dimension 3. *Ann. Global Anal. Geom.* **21** (2002), 111–121.
9. S.-Y. A. Chang, M. Gursky and P. Yang, Prescribing scalar curvature on  $S^2$  and  $S^3$ . *Calc. Var. Partial Differential Equations* **1** (1993), 205–229.
10. S.-Y. A. Chang and P. Yang, A perturbation result in prescribing scalar curvature on  $S^n$ . *Duke Math. J.* **64** (1991), 27–69.
11. S.-Y. A. Chang and P. Yang, Conformal deformation of metrics on  $S^2$ . *J. Differential Geom.* **27** (1988), 259–296.
12. S.-Y. A. Chang and P. Yang, Prescribing Gaussian curvature on  $S^2$ . *Acta Math.* **159** (1987), 215–259.
13. X. Chen and X. Xu, The scalar curvature flow on  $S^n$ —perturbation theorem revisited. *Invent. Math.* **187** (2012), 395–506.
14. J. H. Cheng, Curvature functions for the sphere in pseudohermitian geometry. *Tokyo J. Math.* **14** (1991), 151–163.
15. H. Chtioui, K. Elmehdi and N. Gamara, The Webster scalar curvature problem on the three dimensional CR manifolds. *Bull. Sci. Math.* **131** (2007), 361–374.
16. H. L. Chiu, Compactness of pseudohermitian structures with integral bounds on curvature. *Math. Ann.* **334** (2006), 111–142.
17. B. Chow, The Yamabe flow on locally conformally flat manifolds with positive Ricci curvature. *Comm. Pure Appl. Math.* **45** (1992), 1003–1014.
18. S. Dragomir and G. Tomassini, Differential Geometry and Analysis on CR Manifolds, Progress in Mathematics, **246**, Birkhäuser Boston, Boston, MA, (2006).
19. V. Felli and F. Uguzzoni, Some existence results for the Webster scalar curvature problem in presence of symmetry. *Ann. Math. Pura Appl. (4)* **183** (2004), 469–493.

20. G. Folland and E. Stein, Estimates for the  $\bar{\partial}_b$  complex and analysis on the Heisenberg group. *Comm. Pure Appl. Math.* **27** (1974), 429–522.
21. R. L. Frank and E. H. Lieb, Sharp constants in several inequalities on the Heisenberg group. *Ann. of Math. (2)* **176** (2012), 349–381.
22. N. Gamara, The CR Yamabe conjecture—the case  $n = 1$ . *J. Eur. Math. Soc.* **3** (2001), 105–137.
23. N. Gamara, The prescribed scalar curvature on a 3-dimensional CR manifold. *Adv. Nonlinear Stud.* **2** (2002), 193–235.
24. N. Gamara and R. Yacoub, CR Yamabe conjecture—the conformally flat case. *Pacific J. Math.* **201** (2001), 121–175.
25. R. S. Hamilton, The Ricci flow on surfaces. *Contemp. Math.*, **71**, Amer. Math. Soc., Providence, RI, (1988), 237–262.
26. P. T. Ho, Prescribed curvature flow on surfaces. *Indiana Univ. Math. J.* **60** (2011), 1517–1542.
27. P. T. Ho, Result related to prescribing pseudo-Hermitian scalar curvature. *Int. J. Math.* **24** (2013), 29pp.
28. P. T. Ho, The long time existence and convergence of the CR Yamabe flow. *Commun. Contemp. Math.* **14** (2012), 50 pp.
29. D. Jerison and J.M. Lee, Extremals for the Sobolev inequality on the Heisenberg group and the CR Yamabe problem. *J. Amer. Math. Soc.* **1** (1988), 1–13.
30. D. Jerison and J. M. Lee, Intrinsic CR normal coordinates and the CR Yamabe problem. *J. Differential Geom.* **29** (1989), 303–343.
31. D. Jerison and J. M. Lee, The Yamabe problem on CR manifolds. *J. Differential Geom.* **25** (1987), 167–197.
32. J. Kazdan and F. Warner, Curvature functions for compact 2-manifolds. *Ann. of Math. (2)* **99** (1974), 14–47.
33. J. Kazdan and F. Warner, Existence and conformal deformation of metrics with prescribed Gaussian and scalar curvatures. *Ann. of Math. (2)* **101** (1975), 317–331.
34. J. M. Lee and T. H. Parker, The Yamabe problem. *Bull. Amer. Math. Soc. (N.S.)* **17** (1987), 37–91.
35. A. Malchiodi and F. Uguzzoni, A perturbation result in prescribing Webster curvature on the Heisenberg sphere. *J. Math. Pure. Appl.* **81** (2002), 983–997.
36. M. Riahia and N. Gamara, Multiplicity results for the prescribed Webster scalar curvature on the three CR sphere under “flatness condition”. *Bull. Sci. Math.* **136** (2012), 72–95.
37. A. Sánchez-Calle, Fundamental solutions and geometry of the sum of squares of vector fields. *Invent. Math.* **78** (1984), 143–160.
38. E. Salem and N. Gamara, The Webster scalar curvature revisited: the case of the three dimensional CR sphere. *Calc. Var. Partial Differential Equations* **42** (2011), 107–136.
39. R. Schoen, Conformal deformation of a Riemannian metric to constant scalar curvature. *J. Differential Geom.* **20** (1984), 479–495.
40. H. Schwetlick and M. Struwe, Convergence of the Yamabe flow for “large” energies. *J. Reine Angew. Math.* **562** (2003), 59–100.
41. M. Struwe, A flow approach to Nirenberg’s problem. *Duke Math. J.* **128** (2005), 19–64.
42. N. S. Trudinger, Remarks concerning the conformal deformation of Riemannian structures on compact manifolds. *Ann. Scuola Norm. Sup. Pisa (3)* **22** (1968), 265–274.
43. H. Yamabe, On a deformation of Riemannian structures on compact manifolds. *Osaka Math. J.* **12** (1960), 21–37.
44. R. Ye, Global existence and convergence of Yamabe flow. *J. Differential Geom.* **39** (1994), 35–50.
45. Y. Zhang, The contact Yamabe flow. Ph.D. thesis, University of Hanover (2006).

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